GRESHAM’S LAW OF MODEL AVERAGING

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Abstract.
An agent operating in a self-referential environment thinks the parameters
of his model might be time-varying. In response, he estimates two models, one with
time-varying parameters, and another with constant parameters. Forecasts are then
based on a Bayesian Model Averaging strategy, which mixes forecasts from the two
models. In reality, structural parameters are constant, but the (unknown) true model
features expectational feedback, which the agent’s reduced form models neglect.
This feedback allows the agent’s fears of parameter instability to be self-confirming.
Within the context of a standard linear present value asset pricing model, we use the tools
of large deviations theory to show that the agent’s self-confirming beliefs about parameter
instability exhibit Markov-switching dynamics between periods of tranquility and periods
of instability. However, as feedback increases, the duration of the unstable state increases,
and instability becomes the norm. Even though the constant parameter model would
converge to the (constant parameter) Rational Expectations Equilibrium if considered
in isolation, the mere presence of an unstable alternative drives it out of consideration.
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1. Introduction
Econometric model builders quickly discover their parameter estimates are unstable. It’s not at all clear how to respond to this. Maybe this drift is signalling model mis-
specification. If so, then by appropriately adapting a model’s specification, parameter
drift should dissipate over time. Unfortunately, evidence suggests that drift persists even
when models are adapted in response to the drift. Another possibility is that the un-
derlying environment is inherently and exogenously nonstationary, so there is simply no
hope of describing economic dynamics in models with constant parameters. Clearly, this
is a rather negative prognosis. Our paper considers a new possibility, one that is consis-
tent with both the observed persistence of parameter drift, and its heteroskedastic nature.
We show that in self-referential environments, where the agent’s own beliefs influence the
data-generating process (DGP), it is possible that persistent parameter drift becomes self-
confirming. That is, parameters drift simply because agents think they might drift. We
show that this instability can arise even in models that would have unique and determi-
nate equilibria if parameters were known. Self-confirming volatility arises here because

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See, e.g., Cogley and Sargent (2005), Fernandez-Villaverde and Rubio-Ramirez (2007), and Inoue and
Rossi (2011) for evidence on parameter instability in macroeconomic models. Bacchetta and van Wincoop
(2013) discuss parameter instability in exchange rate models.
agents are assumed to be unaware of their own influence over the DGP, and respond to it indirectly by adapting parameter estimates.\footnote{Another possibility is sometimes advanced, namely, that parameter drift is indicative of the Lucas Critique at work. This is an argument that Lucas (1976) himself made. However, as noted by Sargent (1999), the Lucas Critique (by itself) cannot explain parameter drift.}

We consider a standard present value asset pricing model. This model relates current prices to current fundamentals, and to current expectations of future prices. Agents are assumed to be unaware of this expectational feedback. Instead, they posit reduced form models, and update coefficient estimates as needed. If agents believe parameters are constant, and update estimates accordingly using recursive Least-Squares, their beliefs will eventually converge to the true (constant parameters) Rational Expectations equilibrium (see, e.g., Evans and Honkapohja (2001) for the necessary stability conditions). On the other hand, if they are convinced parameters drift, with a constant innovation variance that is strictly positive, they will estimate parameters using the Kalman filter, and their beliefs will exhibit persistent fluctuations around the Rational Expectations equilibrium.

A large recent literature argues that these so-called ‘constant gain’ (or ‘perpetual learning’) models are useful for understanding a wide variety of dynamic economic phenomena.\footnote{Examples include: Sargent (1999), Cho, Williams, and Sargent (2002), Marcet and Nicolini (2003), Kasa (2004), Chakraborty and Evans (2008), and Benhabib and Dave (2014).} However, several nagging questions plague this literature - Why are agents so convinced that parameters are time-varying? In terms of explaining volatility, don’t these models in a sense “assume the result”? What if agents’ beliefs were less dogmatic, and allowed for the possibility that parameters were constant?

Our paper addresses these questions. It is inspired by the numerical exercise in Evans, Honkapohja, Sargent, and Williams (2013) (henceforth, EHSW). They study a standard cobweb model, in which agents consider two models. One model has constant parameters, and the other has time-varying parameters (TVP). When computing forecasts of next period’s price, agents hedge their bets by engaging in a traditional Bayesian Model Averaging strategy. That is, forecasts are the posterior probability weighted average of the two models’ forecasts. Using simulations, they find that if expectation feedback is sufficiently strong, the weight on the TVP model can converge to 1 (i.e., the agent believes the TVP model is the correct model), even though the underlying structural model features constant parameters. As in Gresham’s Law, ‘bad models can drive out good models’.\footnote{Gresham’s Law is named for Sir Thomas Gresham, who was a financial adviser to Queen Elizabeth I. He is often credited for noting that ‘bad money drives out good money’. Not surprisingly, ‘Gresham’s Law’ is a bit of a misnomer. As DeRoover (1949) documents, it was certainly known before Gresham, with clear descriptions by Copernicus, Oresme, and even Aristophanes. There is also debate about its empirical validity (Rolnick and Weber (1986)).} EHSW assert that the prior belief of the agent satisfies the ‘grain of truth’ condition of Kalai and Lehrer (1993), in the sense that the prior distribution assigns a positive weight to a correctly specified model. It is left as a puzzle that Bayesian learning can apparently converge to a wrong model (i.e., the TVP model), contrary to what Kalai and Lehrer (1993) proved.

We formalize the observations and strengthen the insights of EHSW in a number of important ways. First, we show that the results of Evans, Honkapohja, Sargent, and Williams (2013) hold for a broader class of models, by analyzing an asset pricing model...
rather than the cobweb model analyzed by EHSW. Our analysis applies almost directly to
the cobweb model. Second, we prove analytically that the posterior belief that the TVP
model is the true model converges to 1 with a positive probability: Gresham’s law may
prevail. Along the way, we explain why convergence to 1 can occur only if the expectational
feedback parameter is sufficiently large. Third, if the variance of the parameter innovation
term of TVP model converges to 0, the same posterior belief spends almost all its time in
the neighborhood of 1: Gresham’s law must prevail in the long run.

Finally, and most importantly, we formulate a new solution concept, which generalizes
the notion of a self-confirming equilibrium. In a rational expectations equilibrium, each
player has a perceived law of motion, which is precisely the same as the actual law of
motion. As a result, the perceived law of motion of every player must be the same. In
our case, the perceived law of motion of each player can differ, yet is confirmed by his
observations of the other players’ behavior. Our solution concept, which we term self-
confirmed behavior, consists of a pair of perceived laws of motion and a criterion defining
belief confirmation for each player, in which the behavior induced by the perceived laws of
motion is confirmed by the observed behavior of each player. This requirement is milder
than the requirement that perceived laws of all players must coincide with each other
as in a rational expectations equilibrium. In particular, self-confirming behavior permits
different perceived laws of motion to survive along the equilibrium path, some of which
must be misspecified. In contrast, self-confirming equilibria only permit different per-
ceived laws of motion off the equilibrium path (Fudenberg and Levine (1993)). Thus, with
sufficient experimentation, differences of perception should disappear in a self-confirming
equilibrium. As frequent experiments reveals outcomes off the equilibrium path, some
self-confirming equilibrium may collapse. In contrast, self-confirmed behavior is shown to
be robust against frequent experiments.

In order to analyze the asymptotic properties of Bayesian learning dynamics, we exploit
the fact that variables evolve at different speeds. The data arrive on a relatively fast
calendar time-scale. Estimates of the TVP model evolve on a slower time-scale, determined
by the innovation variance of the parameters. Estimates of the constant-parameter model
evolve even slower, on a time-scale determined by the inverse of the historical sample size.
In the limit, the difference in the speed at which these variables evolve becomes so large
that we can treat them as if they evolve according to different ‘time scales’: we can treat
the estimates of the constant parameter model as ‘fixed,’ when we analyze the asymptotic
properties of the estimates of the TVP model. This hierarchy of time-scales allows us to
exploit two time scale stochastic approximation methods (Borkar (2008)) to analyze the
convergence and stability properties of the two models’ parameter estimates.

We show that for a given variance of the innovation term of the coefficient of the
TVP model, positive weight is assigned to either the constant parameter model or the
TVP model. As the variance of the innovation term vanishes, we compare the escape
probability from the domain of attraction of each limit point. To this end, we compare
the large deviation rate functions around the boundary of the domain of the attraction to
see which limit point is assigned larger weight in the limit, as the variance of the innovation
term vanishes.

We prove that if expectational feedback is sufficiently strong, the weight on the TVP
model converges to one. In this sense, Gresham was right; bad models do indeed drive out
good models. With empirically plausible parameter values, we find that steady state asset
price volatility is more than 90% higher than it would be if agents just used the constant
parameters model.

The intuition for why the TVP model eventually dominates is the following - When
the weight on the TVP model is close to one, the world is relatively volatile (due to
feedback). This makes the constant parameters model perform relatively poorly, since it
is unable to track the feedback-induced time-variation in the data. Of course, the tables
are somewhat turned when the weight on the TVP model is close to zero. Now the world
is relatively tranquil, and the TVP model suffers from additional noise, which puts it at a
competitive disadvantage. However, as long as this noise isn’t too large, the TVP model
can take advantage of its ability to respond to rare sequences of shocks that generate ‘large
deviations’ in the estimates of the constant parameters model. In a sense, during tranquil
times, the TVP model is lying in wait, ready to pounce on, and exploit, large deviation
events. These events provide a foothold for the TVP model, which due to feedback, allows
it to regain its dominance. It is tempting to speculate whether this sort of mechanism
could be one factor in the lingering, long-term effects of rare events like financial crises.

The remainder of the paper is organized as follows. The next section presents our
asset pricing version of the model in EHSW (2013). We first study the implications of
learning with only one model, and discuss whether beliefs converge to self-confirming
equilibria. We then allow the agent to consider both models simultaneously, and examine
the implications of Bayesian Model Averaging. Section 3 contains our proof that the
weight on the TVP model eventually converges to one. Section 4 illustrates our results
with a variety of simulations. These simulations reveal that during the transition the
agent occasionally switches between the two models. Section 5 discusses the robustness
of the results to alternative definitions of the model space, and emphasizes the severity of
Kalai and Lehrer’s (1993) ‘grain of truth’ condition. Finally, the Conclusion discusses a
few extensions and potential applications, while the Appendix collects proofs of various
technical results.

2. To believe is to see

To illustrate the basic idea, we use a simple asset pricing model as our laboratory. This
example is inspired by the numerical simulations of EHSW using a cobweb model. We
argue that the findings of EHSW can potentially apply to a broader class of dynamic
models.

2.1. Description. Consider the following workhorse asset-pricing model, in which an
asset price at time $t$, $p_t$, is determined according to

$$p_t = \delta z_t + \alpha E_t p_{t+1} + \sigma \epsilon_t$$  \hspace{1cm} (2.1)

where $z_t$ denotes observed fundamentals (e.g., dividends), and where $\alpha \in (0,1)$ is a (con-
stant) discount rate, which determines the strength of expectational feedback. Empirically,
it is close to one. The $\epsilon_t$ shock is Gaussian white noise. Fundamentals are assumed to
evolve according to the AR(1) process

$$z_t = \rho z_{t-1} + \sigma z \epsilon_{z,t}$$  \hspace{1cm} (2.2)
for $\rho \in (0, 1)$. The fundamentals shock, $\epsilon_{z,t}$, is Gaussian white noise, and is assumed to be orthogonal to the price shock $\epsilon_t$. The unique stationary rational expectations equilibrium is

$$p_t = \frac{\delta}{1 - \alpha \rho} z_t + \sigma \epsilon_t. \quad (2.3)$$

Along the equilibrium path, the dynamics of $p_t$ can only be explained by the dynamics of fundamentals, $z_t$. Any excess volatility of $p_t$ over the volatility of $z_t$ must be soaked-up by the exogenous shock $\epsilon_t$.

It is well known that Rational Expectations versions of this kind of model cannot explain observed asset price dynamics (Shiller (1989)). Not only are prices excessively volatile, but this volatility comes in recurrent ‘waves’. Practitioners respond to this using reduced form ARCH models. Instead, we try to explain this persistent stochastic volatility by assuming that agents are engaged in a process of Bayesian learning. Of course, the notion that learning might help to explain asset price volatility is hardly new (see, e.g., Timmermann (1996) for an early and influential example). However, early examples were based on least-squares learning, which exhibited asymptotic convergence to the Rational Expectations Equilibrium. This would be fine if volatility appeared to dissipate over time, but as noted earlier, there is no evidence for this. In response, a more recent literature has assumed that agents use so-called constant gain learning, which discounts old data. This keeps learning alive. For example, Benhabib and Dave (2014) show that constant gain learning can generate persistent excess volatility, and can explain why asset prices have fat-tailed distributions even when the distribution of fundamentals is thin-tailed.

Our paper builds on the work of Benhabib and Dave (2014). The key parameter in their analysis is the update gain. Not only do they assume it is bounded away from zero, but they restrict it to be constant. Following Sargent and Williams (2005), they note that a constant gain can provide a good approximation to the (steady state) gain of an optimal Kalman filtering algorithm. However, they go on to show that the learning dynamics exhibit recurrent escapes from this steady state. This calls into question whether agents would in fact cling to a constant gain in the presence of such instability. Here we allow the agent to effectively employ a time-varying gain, which is not restricted to be nonzero. We do this by supposing that agents average between a constant gain and a decreasing/least-squares gain. Evolution of the model probability weights delivers a state-dependent gain. In some respects, our analysis resembles the gain-switching algorithm of Marcet and Nicolini (2003). However, they require the agent to commit to one or the other, whereas we permit the agent to be a Bayesian, and average between the two. Despite the fact that our specification of the gain is somewhat different, like Benhabib and Dave (2014), we rely on the theory of large deviations to provide an analytical characterization of the Markov-switching escape dynamics.

2.2. Learning with a correct model. Suppose the agent knows the fundamentals process in (2.2), but does not know the structural price equation in (2.1). Instead, the agent postulates the following state-space model for prices

$$p_t = \beta_t z_t + \sigma \epsilon_t$$
$$\beta_t = \beta_{t-1} + \sigma_v v_t. \quad (2.4)$$
where it is assumed that \( \text{cov}(\epsilon, v) = 0 \). Note that the Rational Expectations equilibrium is a special case of this, with

\[
\sigma_v = 0 \quad \text{and} \quad \beta = \frac{\delta}{1 - \alpha \rho}.
\]

For now, let us suppose the agent adopts the dogmatic prior that parameters are constant.

\[ \mathcal{M}_0 : \sigma_v^2 = 0. \]

Given this belief, he estimates the unknown parameter of his model using the following Kalman filter algorithm

\[
\beta_{t+1} = \beta_t + \left( \frac{\Sigma_t}{\sigma^2 + \Sigma_t z_t^2} \right) z_t (p_t - \beta_t z_t) \quad (2.5)
\]

\[
\Sigma_{t+1} = \Sigma_t - \frac{(z_t \Sigma_t)^2}{\sigma^2 + \Sigma_t z_t^2} \quad (2.6)
\]

where we adopt the common assumption that \( \beta_t \) is based on time-(\( t - 1 \)) information, while the time-\( t \) forecast of prices, \( \beta_t z_t \), can incorporate the latest \( z_t \) observation. This assumption is made to avoid simultaneity between beliefs and observations. The process, \( \Sigma_t \), represents the agent’s evolving estimate of the variance of \( \beta_t \).

Notice that given his beliefs that parameters are constant, \( \Sigma_t \) converges to zero at rate \( t^{-1} \). This makes sense. If parameters really are constant, then each new observation contributes less and less relative to the existing stock of knowledge. On the other hand, notice that during the transition, the agent’s beliefs are inconsistent with the data. He \textit{thinks} \( \beta \) is constant, but due to expectational feedback, his own learning causes \( \beta \) to be time-varying. This can be seen by substituting the agent’s time-\( t \) forecast into the true model in \((2.1)\)

\[
p_t = [\delta + \rho \alpha \beta_t] z_t + \sigma \epsilon_t
\]

It is fair to say that opinions differ as to whether this inconsistency is important. As long as the \( T \)-mapping between beliefs and outcomes has the appropriate stability properties, the agent’s incorrect beliefs will eventually be corrected. That is, learning-induced parameter variation eventually dissipates, and the agent eventually learns the Rational Expectations equilibrium. However, as pointed out by Bray and Savin (1986), in practice this convergence can be quite slow, and one could then reasonably ask why agents aren’t able to detect the parameter variation that their own learning generates. If they do, wouldn’t they want to revise their learning algorithm, and if they do, will learning still take place?\(^6\)

In our view, this debate is largely academic, since the more serious problem with this model is that it fails to explain the data. Since learning is transitory, so is any learning

\(^5\)See Evans and Honkapohja (2001) for further discussion.

\(^6\)McGough (2003) addresses this issue. He pushes the analysis one step back, and shows that if agents start out with a time-varying parameter learning algorithm, but have priors that this variation damps out over time, then agents can still eventually converge to a constant parameter Rational Expectations equilibrium.
induced parameter instability. Although there is some evidence in favor of a 'Great Moderation' in the volatility of macroeconomic aggregates (at least until the recent financial crisis!), there is little or no evidence for such moderation in asset markets. As a result, more recent work assumes agents view parameter instability as a permanent feature of the environment.

2.3. Learning with a wrong model. Now assume the agent has a different dogmatic prior. Suppose he is now convinced that parameters are time-varying, which can be expressed as the parameter restriction

\[ M_1 : \sigma_v^2 > 0. \]

Although this is a 'wrong model' from the perspective of the (unknown) Rational Expectations equilibrium, the more serious specification error here is that the agent does not even entertain the possibility that parameters might be constant. This prevents him from ever learning the Rational Expectations equilibrium (Bullard (1992)). Still, due to feedback, there is a sense in which his beliefs about parameter instability can be self-confirming, since ongoing belief revisions will produce ongoing parameter instability.

The belief that \( \sigma_v^2 > 0 \) produces only a minor change in the Kalman filtering algorithm in (2.5) and (2.6). We just need to replace the Riccati equation in (2.6) with the new Riccati equation

\[ \Sigma_{t+1} = \Sigma_t - \frac{(z_t \Sigma_t)^2}{\sigma^2 + \Sigma_t z_t^2} + \sigma_v^2 \]  

(2.7)

The additional \( \sigma_v^2 \) term causes \( \Sigma_t \) to now converge to a strictly positive limit, \( \Sigma > 0 \). As noted by Benveniste et. al. (1990, pgs. 139-40), if we assume \( \sigma_v^2 \ll \sigma^2 \), which we will do in what follows, we can use the approximation \( \sigma^2 + \Sigma_t z_t^2 \approx \sigma^2 \) in the above formulas (\( \Sigma_t \) is small relative to \( \sigma^2 \) and scales inversely with \( z_t^2 \)). The Riccati equation in (2.7) then delivers the following approximation for the steady state variance of the state, \( \Sigma \approx \sigma \cdot \sigma_v M_z^{-1/2} \), where \( M_z = \mathbb{E}(z_t^2) \) denotes the second moment of the fundamentals process. In addition, if we further assume that priors about parameter drift take the particular form, \( \sigma_v^2 = \gamma^2 \sigma^2 M_z^{-1} \), then the steady state Kalman filter takes the form of the following (discounted) recursive least-squares algorithm

\[ \beta_{t+1} = \beta_t + \gamma M_z^{-1} z_t (p_t - \beta_t z_t) \]  

(2.8)

where the agent’s priors about parameter instability are now captured by the so-called ‘gain’ parameter, \( \gamma \). If the agent thinks parameters are more unstable, he will use a higher gain.

Constant gain learning algorithms have explained a wide variety of dynamic economic phenomena. For example, Cho, Williams, and Sargent (2002) show they potentially explain US inflation dynamics. Kasa (2004) argues they can explain recurrent currency crises. Chakraborty and Evans (2008) show they can explain observed biases in forward exchange rates, while Benhabib and Dave (2014) show they explain fat tails in asset price distributions.
An important question raised by this literature arises from the fact that the agent’s model is ‘wrong’. Wouldn’t a smart agent eventually discover this? On the one hand, this is an easy question to answer. Since his prior dogmatically rules out the ‘right’ constant parameter model, there is simply no way the agent can ever detect his misspecification, even with an infinite sample. On the other hand, due to the presence of expectational feedback, a more subtle question is whether the agent’s beliefs about parameter instability can become ‘self-confirming’ (Sargent (2008))? That is, to what extent are the random walk priors in (2.4) consistent with the observed behavior of the parameters in the agent’s model? Would an agent have an incentive to revise his prior in light of the data that are themselves (partially) generated by those priors?

It is useful to divide this question into two pieces, one related to the innovation variance, $\sigma_v^2$, and the other to the random walk nature of the dynamics. As noted above, the innovation variance is reflected in the magnitude of the gain parameter. Typically the gain is treated as a free parameter, and is calibrated to match some feature of the data. However, as noted by Sargent (1999, chpt. 6), in self-referential models the gain should not be treated as a free parameter. It is an equilibrium object. This is because the optimal gain depends on the volatility of the data, but at the same time, the volatility of the data depends on the gain. Evidently, as in a Rational Expectation Equilibrium, we need a fixed point.

In a prescient paper, Evans and Honkapohja (1993) addressed the problem of computing this fixed point. They posed the problem as one of computing a Nash equilibrium. In particular, they ask - Suppose everyone else is using a given gain parameter, so that the data-generating process is consistent with this gain. Would an individual agent have an incentive to switch to a different gain? Under appropriate stability conditions, one can then compute the equilibrium gain by iterating on a best response mapping as usual. Evans and Ramey (2006) extend the work of Evans and Honkapohja (1993). They propose a Recursive Prediction Error algorithm, and show that it does a good job tracking the optimal gain in real-time. They also point out that due to forecast externalities, the Nash gain is typically Pareto suboptimal. More recently, Kostyshyna (2012) uses Kushner and Yang’s (1995) adaptive gain algorithm to revisit the same hyperinflation episodes studied by Marce and Nicolini (2003). The idea here is to recursively update the gain in exactly the same way that parameters are updated. The only difference is that now there is a constant gain governing the evolution of the parameter update gain. Kostyshyna (2012) shows that her algorithm performs better than the discrete, markov-switching algorithm of Marce and Nicolini (2003). In sum, $\sigma_v^2$ can indeed become self-confirming, and agents can use a variety of algorithms to estimate it.

To address the second issue we need to study the dynamics of the agent’s parameter estimation algorithm in (2.8). After substituting in the actual price process this can be written as

$$\beta_{t+1} = \beta_t + \gamma M^{-1} z_t \{[\delta + (\alpha p - 1)\beta_t] z_t + \sigma \epsilon_t\}$$

(2.9)

7Of course, a constant gain model could be the ‘right’ model too, if the underlying environment features exogenously time-varying parameters. After all, it is this possibility that motivates their use in the first place. Interestingly, however, most existing applications of constant gain learning feature environments in which doubts about parameter stability are entirely in the head of the agents.
Let $\beta^* = \delta/(1 - \alpha \rho)$ denote the Rational Expectations equilibrium. Also let $\tau_t = t \cdot \gamma$, and then define $\beta(\tau_t) = \beta_t$. We can then form the piecewise-constant continuous-time interpolation, $\beta(\tau) = \beta(\tau_t)$ for $\tau \in [t \gamma, t \gamma + \gamma)$. Although for a fixed $\gamma$ (and $\sigma_v^2$) the paths of $\beta(\tau)$ are not continuous, they converge to the following continuous limit as $\sigma_v^2 \to 0$ (see Evans and Honkapohja (2001) for a proof)

**Proposition 2.1.** As $\sigma_v^2 \to 0$, $\beta(\tau)$ converges weakly to the solution of the following diffusion process

$$d\beta = -(1 - \alpha \rho)(\beta - \beta^*)d\tau + \gamma M^{-1/2}_z \sigma dW_{\tau}$$

(2.10)

where $dW_{\tau}$ is the standard Wiener process.

This is an Ornstein-Uhlenbeck process, which generates a stationary Gaussian distribution centered on the Rational Expectations equilibrium, $\beta^*$. Notice that the innovation variance is consistent with the agent’s priors, since $\gamma^2 \sigma^2 M_z^{-1} = \sigma_v^2$. However, notice also that $d\beta$ is autocorrelated. That is, $\beta$ does not follow a random walk. Strictly speaking then, the agent’s priors are misspecified. However, remember that traditional definitions of self-confirming equilibria presume that agents have access to infinite samples. In practice, agents only have access to finite samples. Given this, we can ask whether the agent could statistically reject his prior. This will be difficult when the drift in (2.10) is small. This is the case when: (1) Estimates are close to the $\beta^*$, (2) Fundamentals are persistent, so that $\rho \approx 1$, and (3) Feedback is strong, so that $\alpha \approx 1$.

These results show that if the ‘grain of truth’ assumption fails, wrong beliefs can be quite persistent (Esponda and Pouzo (2014)). One might argue that if the agent were to expand his priors to include $M_0$, the correct model would eventually dominate the wrong model in the sense that the posterior assigned to the correct model converges to 1 (Kalai and Lehrer (1993)).

We claim otherwise, and demonstrate that the problem arising from the presence of misspecified models can be far more insidious. We do this by expanding upon the example presented by Evans, Honkapohja, Sargent, and Williams (2013). The mere presence of a misspecified alternative can disrupt the learning process.

### 2.4. Model Averaging

Dogmatic priors (about anything) are rarely a good idea. So let us now suppose the agent hedges his bets by entertaining the possibility that parameters are constant. Forecasts are then constructed using a traditional Bayesian Model Averaging (BMA) strategy. This strategy effectively ‘convexifies’ the model space. If we let $\pi_t$ denote the current probability assigned to $M_1$, the TVP model, and let $\beta_t(i)$ denote the current parameter estimate for $M_i$ ($i = 0, 1$), the agent’s time-$t$ forecast becomes

$$E_{t} p_{t+1} = \rho [\pi_t \beta_t(1) + (1 - \pi_t) \beta_t(0)] z_t$$

which implies that the actual law of motion for price is

$$p_t = (\delta + \alpha \rho [\pi_t \beta_t(1) + (1 - \pi_t) \beta_t(0)]) z_t + \sigma \epsilon_t.$$  (2.11)

Observing $p_t$, each forecaster updates his belief about $\beta_t(i)$ ($i \in \{0, 1\}$), and the decision maker updates $\pi_t$. Our task is to analyze the asymptotic properties of $(\beta_t(1), \beta_t(0), \pi_t)$.  

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8In the language of Hansen and Sargent (2008), we can compute the detection error probability.

9To ease notation in what follows, we shall henceforth omit the hats from the parameter estimates.
Since each component in \((\beta_t(1), \beta_t(0), \pi_t)\) evolves at a different “speed,” it would be useful to define the notion of “speed” in terms of the time scale. We use the sample average time scale, \(1/t\), as the benchmark. This is the time-scale at which \(\beta_t(0)\) evolves. More precisely, \(\forall \tau > 0\), we can find the unique integer satisfying
\[
\sum_{k=1}^{K-1} \frac{1}{k} < \tau < \sum_{k=1}^{K} \frac{1}{k}.
\]
Let \(m(\tau) = K\) and define
\[
t_K = \sum_{k=1}^{K} \frac{1}{k}.
\]
Therefore, \(t_K \to \infty\) as \(K \to \infty\). We are interested in the sample paths over the tail interval \([t_K, t_K + \tau)\). By the same reason, we are interested in the speed of evolution at the right hand tail of a stochastic process.

**Definition 2.2.** Let \(\varphi_t\) be a stochastic process. We say that \(\varphi_t\) evolves at a faster time scale than \(\beta_t(0)\) if
\[
\lim_{t \to \infty} t|\varphi_t - \varphi_{t-1}| = \infty,
\]
with probability 1, and evolves at a slower time scale than \(\beta_t(0)\) if
\[
\lim_{t \to \infty} t|\varphi_t - \varphi_{t-1}| = 0
\]
with probability 1.

Since \(\sigma^2_v > 0\), \(\beta_t(1)\) evolves at the speed of a constant gain algorithm
\[
\lim_{t \to \infty} t|\beta_t(1) - \beta_{t-1}(1)| = \infty
\]
with probability 1.

### 3. Self-Confirming Behavior

Our model features three players, each of whom is endowed with a different perceived law of motion. We need to spell out the perceived law of motion of each player. The perception of a player has two components: first, each forecaster has a perceived law of motion of his model’s parameters, and second, he has a perceived law of motion of the behavior of the other players.

There are three types of players in the model: Two competing forecasters (\(M_1\) and \(M_0\)), and a policymaker who combines the two forecasts to determine the actual price according to his posterior \(\pi_t\) that \(M_1\) is the correct model. All three players correctly perceive that the actual price is determined as a convex combination of the forecasts from \(M_1\) and \(M_0\), and that the weight assigned to the \(M_1\) forecast is \(\pi_t\). However, the perceived law of motion of each player may be incorrect (or misspecified). This influences the behavior of each player substantially.

#### 3.1. Perceptions of Each Player.
All players know that \(z_t\) evolves according to (2.2). They also know that the actual law of motion of \(p_t\) is (2.11).
3.1.1. **Perception of** \(M_0\) **Forecaster.** The \(M_0\) forecaster thinks \(\beta(0)\) is an unobserved constant. At the same time, he also thinks \(\beta_t(1)\) is a random variable with a fixed mean:

\[
\beta_t(1) = \tilde{\beta}(1) + \xi_t
\]

(3.12)

where \(\tilde{\beta}(1)\) is an unobserved constant, and \(\xi_t\) is a Gaussian white noise with

\[
\mathbb{E} \xi_t^2 = \Sigma^*(1).
\]

(3.13)

The \(M_0\) forecaster also believes that \(\beta_t(1)\) is selected optimally:

\[
\lim_{t \to \infty} \mathbb{E} \left[ p_t - (\delta z_t + \alpha \rho \tilde{\beta}(1) z_t) \right] = 0.
\]

For simplicity, let us assume that the \(M_0\) forecaster has a dogmatic belief about \(\Sigma^*(1)\) and consequently, does not bother to estimate \(\Sigma^*(1)\). Note that (3.12) will differ from the perceived law of motion of the \(M_1\) forecaster. In this sense, his perceived law of motion is misspecified. Finally, the \(M_0\) forecaster correctly perceives that \(\pi_t\) is an endogenous variable, but believes that \(\pi_t\) evolves on a slower time scale than \(\beta_t(0)\).

3.1.2. **Perception of** \(M_1\) **Forecaster.** The \(M_1\) forecaster thinks the parameters of his model drift

\[
\beta_t(1) = \beta_{t-1}(1) + \epsilon_{v,t}
\]

(3.14)

where \(\epsilon_{v,t}\) is a Gaussian white noise with \(\mathbb{E} \epsilon_{v,t}^2 = \sigma_v^2\). This is also a misspecified belief, in the sense that the unique rational expectations equilibrium of this model features constant parameters, if this were known with certainty. To simplify the analysis, we assign a dogmatic belief to the \(M_1\) forecaster about \(\sigma_v^2\), under which he does not attempt to estimate \(\sigma_v^2\) directly, or check whether his belief about \(\sigma_v^2 > 0\) is accurate. The \(M_1\) forecaster presumes that \(\beta_t(0)\) and \(\pi_t\) evolve at a slower time scale than \(\beta_t(1)\).

3.1.3. **Perception of the Decision Maker.** The decision maker is the only fully rational player in our model. He knows the perceived laws of motion of the \(M_1\) and \(M_0\). He is also aware of the (possibly misspecified) model of each player, and correctly forecasts their behavior. He updates \(\pi_t\) according to the Bayes rule.

3.2. **Solution Concept.** Notice that the perceived law of motion of each forecaster does not incorporate the perceived law of motion of the other player. In particular, the \(M_0\) forecaster’s perceived law of motion of the behavior of \(M_1\) forecaster is not the same as \(M_1\) forecaster’s perceived law of \(\beta_t(1)\). This is precisely the point of departure from a rational expectations equilibrium, in which the perceptions of each player must coincide with each other.

**Remark 3.1.** We assume that each player has a dogmatic belief about the evolution of the coefficient he is estimating, only to simplify the analysis. Under the dogmatic belief about (3.14), the \(M_1\) forecaster does not attempt to estimate \(\sigma_v^2\), or to check whether his belief is correct. We also assume that the \(M_0\) forecaster has dogmatic beliefs about (3.13). These two assumptions are only made to simplify the analysis, by focusing on the discrepancy between the perceptions on the behavior of the other agents as the source of bounded rationality. We later discuss how to recover (or estimate) \(\sigma_v^2 > 0\) or \(\Sigma^* > 0\).

\[\text{(3.12)}\]

This belief is needed mainly to justify his belief that \(\tilde{\beta}(1)\) is stationary.
Since (3.14) is in the mind of $\mathcal{M}_1$ forecaster, it is not at all clear whether he can find a sensible estimator of $\sigma^2_v > 0$ whose value is uniformly bounded away from 0. The main point of the discussion is whether we can devise a reasonable estimator which converges to positive values of the second moments, especially $\sigma^2_v > 0$. Note that whether or not an estimation method is sensible depends upon the perceived law of motion of the estimating agent.

The important question is whether this gap between perception and behavior can survive a long series of equilibrium path observations. As in a self-confirming equilibrium, we require that perceptions are consistent with observations asymptotically.

Let $P_i(h_{i,t})$ be the perception of player $i \in \{0, 1, d\}$ at $t$, where $d$ denotes the decision maker, which means a joint probability distribution over $(\beta_t(1), \beta_t(0), \pi_t, z_t)$, conditioned on information $h_{i,t}$. Similarly, let $A(h_t)$ be the actual probability distribution over the state. Note that $h_{i,t}$ includes his own perception about the state, but does not include perceptions of the other players over the state.

Given $P_i(h_{i,t})$ in period $t$, player $i$ chooses an action according to $b_i(P_i(h_{i,t}))$. Along with the evolution of $z_t$ and a random shock, $b_i(\cdot) (i \in \{0, 1, d\})$ determines the actual distribution $A(h_{t+1})$ in period $t + 1$. The evolution of $P_i(h_{i,t+1})$ from $P_i(h_{i,t})$ is governed by the learning behavior of player $i$.

Let $d_i(\cdot, \cdot : h_{i,t})$ be the criterion used by player $i$, to measure the distance between the perceived and actual laws of motion, conditioned on $h_{i,t}$. The choice of $d_i$ depends upon the learning algorithm, and possibly, the specific variables of interest. $d_i$ need not be a metric, but must be 0, if $P_i = A_i$, and positive, if $P_i \neq A_i$.

**Definition 3.2.** A profile of $(P_i, d_i)$ for $i \in \{0, 1, d\}$ is a self-confirming behavior with respect to $d_i$, if $\forall i,$

$$\lim_{t \to \infty} d_i(P_i(h_{i,t}), A_i(h_t) : h_{i,t}) = 0$$

where $h_t$ is induced by $A_i$.

**Example 3.3.** If $P_i(h_{i,t})$ coincides with the actual law of motion conditioned on $h_t$, and $d_i$ is relative entropy, then self-confirming behavior coincides with behavior in a rational expectations equilibrium. Although relative entropy is not a metric, it converges to 0 if and only if the two distributions match perfectly. Otherwise, it is positive.

**Example 3.4.** Consider a model with only the $M_0$ forecaster, whose perceived law of motion $P_0$ presumes that $\beta(0)$ is an unobserved constant. If $d_0$ is the Euclidean metric to measure the distance of the mean and the variance of the actual to those of the perceived distribution, then $P_0$ and $d_0$ generate self-confirming behavior.

### 3.3. Self-confirming Equilibrium vs. Self-confirming Behavior

The formal definition and the two example may not show the fundamental difference between a self-confirming equilibrium and self-confirming behavior. Both solution concepts admit wrong beliefs in equilibrium. In a self-confirming equilibrium, players can disagree about outcomes off the equilibrium path (or an event which is not realized with a positive probability). However, along the equilibrium path, all players must agree upon the actual law of motion. In contrast, players may continue to have different perceived laws of motion along the equilibrium path with self-confirming behavior.
The difference arises from the restriction we imposed on each player about how they see the actual law. In a self-confirming equilibrium, each player forms beliefs along the equilibrium path rationally, applying Bayes rule to an accurate belief. In a self-confirming behavior, each player sees the outcome through the filter of $P_i$. As a result, the same action can be interpreted differently, depending upon perceived law $P_i$. We only require that the perceived law is confirmed by the observed behavior of the other players, thus admitting the possibility that $P_i$ can be different from both the perceived laws of other players and the actual laws.

With sufficient experimentation, we can ensure that all histories are realized with positive probability. Then, a self-confirming equilibrium becomes a rational expectations equilibrium, and every player has the same perceived law, which coincides with the actual law of motion. A self-confirming behavior is immune to frequent experiments. Even along the equilibrium path, different players may have different perceived laws of motion, some of which must be misspecified. In a certain sense, a misspecified model is confirmed by the behavior of other players, which renders a misspecified model extremely resilient.

3.4. Misspecification. Our notion of a misspecified model is slightly different from the usual meaning that a model omits a variable that is present in the true data generating process. To illuminate the difference, let us first consider the decision maker’s model. Since the decision maker is Bayesian, he updates $\pi_t$ according to Bayes rule, starting from a given prior $\pi_0 \in (0, 1)$. Since we are now assuming $\pi_0 > 0$, the agent assigns a positive weight to the constant parameter model, $M_0$, which contains the Rational Expectations equilibrium, $\beta^* = \alpha/(1 - \rho\alpha)$ in the support of the prior over $M_0$. Since $M_1$ is assigned positive probability, the agent’s model is misspecified. The model is misspecified, not because it excludes a variable, but because it includes a variable which is not in the true model (i.e., $\sigma_v^2 > 0$). Normally, in situations where the data-generating process is exogenous, starting from a larger model is rather innocuous. The data will show that any variables not in the true model are insignificant. We shall now see that this is no longer the case when the data-generating process is endogenous.

4. Model Averaging Dynamics

Self-confirming behavior induces a probability distribution over the set of sample paths of $(\beta_t(1), \beta_t(0), \pi_t)$. We are interested in the asymptotic properties of the sample path induced by a self-confirming behavior.

**Theorem 4.1.** If $\{(\beta_t(1), \beta_t(0), \pi_t)\}$ is a sample path induced by a self-confirming behavior, it converges to either $(\beta(1), \beta(0), \pi) = (\delta/(1 - \alpha\rho), \delta/(1 - \alpha\rho), 0)$ or $(\beta(1), \beta(0), \pi) = (\delta/(1 - \alpha\rho), \delta/(1 - \alpha\rho), 0)$.

11Based upon this reasoning, Evans, Honkapohja, Sargent, and Williams (2013) asserted that his prior therefore satisfies the grain of truth condition. See page 100 of Evans, Honkapohja, Sargent, and Williams (2013).

More precisely, we study a setting in which the pair of models used by agents includes a ‘grain of truth’ in the sense that the functional form of one of the models is consistent with the REE of the economy while the other model is misspecified relative to the REE.
\((\delta/(1 - \alpha \rho), \delta/(1 - \alpha \rho), 1)\). The convergence of \(\beta_t(1)\) is defined in terms of weak convergence, while the convergence of the remaining two components is in terms of convergence with probability 1.

Before proving the theorem, we need a series of new concepts and preliminary results.

4.1. Odds ratio. The decision maker updates his prior \(\pi_t = \pi_t(1)\) that the data are generated according to \(\mathcal{M}_1\). After a long tedious calculation, the Bayesian updating scheme for \(\pi_t\) can be written as (see EHSW for a partial derivation)

\[
\frac{1}{\pi_{t+1}} - 1 = \frac{A_{t+1}(0)}{A_{t+1}(1)} \left( \frac{1}{\pi_t} - 1 \right) \tag{4.15}
\]

where

\[
A_t(i) = \frac{1}{\sqrt{\text{MSE}(i)}} e^{-\frac{(p_t - \beta_t(i) z_t)^2}{2 \text{MSE}(i)}}
\]

is the time-\(t\) predictive likelihood function for model \(\mathcal{M}_i\), where

\[
\text{MSE}(i) = E(p_t - \beta_t(i) z_t)^2
\]

is the mean squared forecasting error of \(\mathcal{M}_i\).

To study the dynamics of \(\pi_t\) it is useful to rewrite (4.15) as follows

\[
\pi_{t+1} = \pi_t + \pi_t(1 - \pi_t) \left[ \frac{A_{t+1}(1)/A_{t+1}(0) - 1}{1 + \pi_t(A_{t+1}(1)/A_{t+1}(0) - 1)} \right] \tag{4.16}
\]

which has the familiar form of a discrete-time replicator equation, with a stochastic, state-dependent, fitness function determined by the likelihood ratio. Equation (4.16) reveals a lot about the model averaging dynamics. First, it is clear that the boundary points \(\pi = \{0, 1\}\) are trivially stable fixed points, since they are absorbing. Second, we can also see that there could be an interior fixed point, where \(E(A_{t+1}(1)/A_{t+1}(0)) = 1\). Later, we shall see that this occurs when \(\pi = \frac{1}{2\alpha \rho}\), which is interior if feedback is strong enough (i.e., if \(\alpha > \frac{1}{2\rho}\)). However, we shall also see there that this fixed point is unstable. So we know already that \(\pi_t\) will spend most of its time near the boundary points. This will become apparent when we turn to the simulations. One remaining issue is whether \(\pi_t\) could ever become absorbed at one of the boundary points.

**Proposition 4.2.** As long as the likelihoods of \(\mathcal{M}_0\) and \(\mathcal{M}_1\) have full support, the boundary points \(\pi_t = \{0, 1\}\) are unattainable in finite time.

**Proof.** See Appendix A. \(\Box\)

Since the distributions here are assumed to be Gaussian, they obviously have full support, so Proposition 4.2 applies. Although the boundary points are unattainable, the replicator equation for \(\pi_t\) in (4.16) makes it clear that \(\pi_t\) will spend most of its time near these boundary points, since the relationship between \(\pi_t\) and \(\pi_{t+1}\) has the familiar logit function shape, which flattens out near the boundaries. As a result, \(\pi_t\) evolves very slowly near the boundary points.
It is more convenient to consider the log odds ratio. Let us initialize the likelihood ratio at the prior odds ratio:

\[
\frac{A_0(0)}{A_0(1)} = \frac{\pi_0(0)}{\pi_0(1)}
\]

By iteration we get

\[
\frac{\pi_{t+1}(0)}{\pi_{t+1}(1)} = 1 - \frac{1}{\prod_{k=0}^{t+1} \frac{A_k(0)}{A_k(1)}}.
\]

Taking logs and dividing by \((t + 1)\),

\[
\frac{1}{t + 1} \ln \left( 1 - \frac{1}{\pi_{t+1}} \right) = \frac{1}{t + 1} \sum_{k=0}^{t+1} \ln \frac{A_k(0)}{A_k(1)}.
\]

Now define the average log odds ratio, \(\phi_t\), as follows

\[
\phi_t = \frac{1}{t} \ln \left( 1 - \frac{1}{\pi_t} \right) = \frac{1}{t} \ln \left( \frac{\pi_t(0)}{\pi_t(1)} \right)
\]

which can be written recursively as the following stochastic approximation algorithm

\[
\phi_t = \phi_{t-1} + \frac{1}{t} \left[ \ln \frac{A_t(0)}{A_t(1)} - \phi_{t-1} \right].
\]

Invoking well knowing results from stochastic approximation, we know that the asymptotic properties of \(\phi_t\) are determined by the stability properties of the following ODE

\[
\dot{\phi} = E \left[ \ln \frac{A_t(0)}{A_t(1)} \right] - \phi
\]

which has a unique stable point

\[
\phi^* = E \ln \frac{A_t(0)}{A_t(1)}.
\]

Note that if \(\phi^* > 0\), \(\pi_t \to 0\), while if \(\phi^* < 0\), \(\pi_t \to 1\). The focus of the ensuing analysis is to identify the conditions under which \(\pi_t\) converges to 1, or 0. Thus, the sign of \(\phi^*\), rather than its value, is an important object of investigation.

We shall now show that \(\pi_t\) evolves even more slowly than the \(t^{-1}\) time-scale of \(\beta_t(0)\) and \(\phi_t\). This means that when studying the dynamics of the coefficient estimates near the boundaries, we can treat \(\pi_t\) as fixed. In order to make the notion of “more slowly” precise, we need to define precisely the time scale.

5. Proof of Theorem 4.1

5.1. \(M_1\). Both \(M_1\) and \(M_0\) perceive that \(\pi_t\) evolves on a slower time scale so that it can be treated as a constant. Under this (yet to be proved) hypothesis, each forecaster treat

\[
\pi_t = \pi
\]

for some constant \(\pi\). We shall show that \(\pi\) cannot be in the interior of \([0, 1]\) with positive probability. And, if \(\pi_t \to 0\) or 1, then the evolution of \(\pi_t\) is slower than the evolution of \(\beta_t(0)\) and \(\beta_t(1)\).
Let $p_t^i(i)$ be the period-$t$ price forecast by model $i$,
\[ p_t^i(i) = \beta_t(i)z_t. \]

Since
\[ p_t = \alpha \rho [(1 - \pi_t)\beta_t(0) + \pi_t\beta_t(1)]z_t + \delta z_t + \sigma \epsilon_t, \]
the forecast error of $M_1$ is
\[ p_t - p_t^e(1) = \alpha \rho [(1 - \pi_t)\beta_t(0) + (\alpha \rho \pi_t - 1)\beta_t(1) + \delta]z_t + \sigma \epsilon_t. \]

The $M_1$ forecaster believes that $\beta_t(0)$ evolves on a time scale slower than $\beta_t(1)$, and $\pi_t$ evolves at a time scale slower than $\beta_t(0)$. While the actual Bayesian updating formula for $\beta_t(1)$ is quite complex, the asymptotic properties of $\beta_t(1)$ can be approximated by (2.5), where $\beta_t(0)$ and $\pi_t$ are regarded as fixed.\(^{12}\) For the rest of the analysis, we assume that $\beta_t(1)$ evolves according to (2.5),
\[ \lim_{t \to \infty} E\left[ \alpha \rho (1 - \pi_t)\beta_t(0) + (\alpha \rho \pi_t - 1)\beta_t(1) + \delta \right] z_t + \sigma \epsilon_t = 0 \]
in any limit point of the Bayesian learning dynamics.\(^{13}\) Since $\beta_t(1)$ evolves at a faster rate than $\beta_t(0)$, we can treat $\beta_t(0)$ as a constant. We treat $\pi_t$ as constant also. Define
\[ \overline{\beta}(1) = \lim_{t \to 0} E\beta_t(1) \]
whose value is conditioned on $\pi_t$ and $\beta_t(0)$. Since
\[ \lim_{\Sigma_t(1) \to 0, t \to 0} \left[ \alpha \rho (1 - \pi_t)\beta_t(0) + (\alpha \rho \pi_t - 1)\beta_t(1) + \delta \right] + E(\alpha \rho \pi_t - 1)(\beta_t(1) - \overline{\beta}(1)) = 0. \]

**Proposition 5.1.** Suppose that $\alpha \rho < 1$. For a fixed $\pi \in [0, 1]$, $t \to \infty$, $\beta_t(1)$ converges to a stationary distribution with mean
\[ \frac{\delta + \alpha \rho (1 - \pi)\beta(0)}{1 - \alpha \rho \pi} \]
and a variance which vanishes as $\sigma_v^2$ vanishes.

**Proof.** Apply the weak convergence theorem. \(\square\)

Thus,
\[ \overline{\beta}_t(1) = \frac{\alpha \rho (1 - \pi_t)\beta_t(0) + \delta}{1 - \alpha \rho \pi_t}. \]

Define the deviation from the long-run mean as
\[ \xi_t = \beta_t(1) - \overline{\beta}_t(1). \]

Note that
\[ E_t \xi_t^2 = \Sigma_t(1) \]

The mean squared forecast error of $M_1$ given $(\pi_t, \beta_t(0))$ is then
\[ E_t(p_t - p_t^e(1))^2 = (\alpha \rho \pi_t - 1)^2 \sigma_v^2 \Sigma_t(1) + \sigma^2. \]

\(^{12}\)We shall later show that $\beta_t(0)$ and $\pi_t$ evolve at a slower time scale than $\beta_t(1)$. Thus, $M_1$ forecaster’s belief is confirmed.

\(^{13}\)Existence is implied by the tightness of the underlying space.
and the Bayesian updating rule for $\beta_t(1)$ is
\[
\beta_t(1) = \beta_{t-1}(1) + \frac{\Sigma_{t-1}(1)z_t}{\text{MSE}_t(1)} (p_t - \beta_{t-1}(1)z_t)
\]
\[
\Sigma_t(1) = \Sigma_{t-1}(0) - \frac{\Sigma_{t-1}(1)z_t^2}{\text{MSE}_t(1)} + \sigma_v^2
\]
where
\[
\text{MSE}_t(1) = (1 - \alpha \rho \pi_t)^2 \sigma_z^2 \Sigma_{t-1}(1) + \Sigma_{t-1}(0) \sigma_z^2 + \sigma^2. \tag{5.17}
\]

5.2. $M_0$. The $M_0$ forecaster believes in the constant parameter model, and regards $\pi_t$ as a known constant, since it evolves on a slower time scale than $\beta_t(0)$. Moreover, under the assumption that $\beta_t(1)$ has a stationary distribution with a “fixed” mean, the evolution of the conditional mean of the posterior about $\beta(0)$ can be approximated by
\[
\beta_t(0) = \beta_{t-1}(0) + \frac{\Sigma_{t-1}(0)z_t}{\text{MSE}_t^*(0)} (p_t - \beta_{t-1}(1)z_t)
\]
\[
\Sigma_t(0) = \Sigma_{t-1}(0) - \frac{\Sigma_{t-1}(0)z_t^2}{\text{MSE}_t^*(0)}
\]
where
\[
\text{MSE}_t^*(0) = (1 - \alpha \rho (1 - \pi_t))^2 \Sigma_{t}(0) \sigma_z^2 + \alpha^2 \rho^2 \pi_t^2 \Sigma^*(1) \sigma_z^2 + \Sigma_{t-1}(0) \sigma_z^2 + \sigma^2. \tag{5.18}
\]
By being “fixed,” we mean that $M_0$ forecaster perceives $\beta_t(1)$ is fixed, because $M_0$ forecaster perceives $\beta_t(0)$ is a fixed coefficient, and $\beta_t(1)$ is a function of $\beta_t(0)$. Note that instead of $\Sigma_t(1)$, (5.18) has $\Sigma^*(1)$ as defined in (3.13). That is because $M_0$ forecaster has a dogmatic belief that $\beta_t(1)$ has a stationary distribution, with variance $\Sigma^*(1)$.

Since the $M_0$ forecaster thinks the state variables are stationary, $\beta_t(0)$ evolves on a slower time scale than $\beta_t(1)$, so that the perception of $M_1$ about the behavior of $\beta_t(0)$ is confirmed. After $\beta_t(1)$ reaches its own limit for a fixed $(\pi_t, \beta_t(0))$, we calculate the asymptotic behavior of $\beta_t(0)$. Define
\[
\bar{\beta}(0) = \lim_{t \to \infty} \beta_t(0).
\]
We can calculate the associated ODE of $\beta_t(0)$.
\[
\dot{\beta}(0) = \delta + \alpha \rho (\pi \bar{\beta}(1) + (1 - \pi) \bar{\beta}(0)) - \bar{\beta}(0),
\]
Since
\[
E \left[ \beta_t(1) - \frac{\alpha \rho (1 - \pi_t) \beta_t(0) + \delta}{1 - \alpha \rho \pi_t} \right] = 0,
\]
\[
\bar{\beta}(1) = \frac{\alpha \rho (1 - \pi) \beta(0) + \delta}{1 - \alpha \rho \pi}.
\]
The unique stable stationary point of the associated ODE is
\[
\bar{\beta}(0) = \bar{\beta}(1) = \frac{\delta}{1 - \alpha \rho}.
\]
Define
\[
\text{MSE}(1) = \lim_{t \to \infty} \text{MSE}_t(1)
\]
and

\[ \text{MSE}^*(0) = \lim_{t \to \infty} \text{MSE}^*_t(0). \]

One can show

\[ \text{MSE}(1) = (1 - \alpha \rho \pi)^2 \Sigma \sigma_z^2 + \sigma^2 \]

\[ \text{MSE}^*(0) = (\alpha \rho \pi)^2 \Sigma^* (1) \sigma_z^2 + \sigma^2. \]

As \( \beta_t(1) \) evolves at a faster time scale than \( \beta_t(0) \), \( M_0 \) forecaster perceives that \( \beta_t(0) \) distributes according to the stationary distribution. As his belief that the state remains stationary is self-confirmed, \( M_0 \) forecaster has reason to use the decreasing gain algorithm, which in turn confirms the belief of \( M_1 \) forecaster about the behavior of \( M_0 \) forecaster.

5.3. \( \pi_t \). Since the Kalman gain for the recursive formula for \( \beta_t(1) \) is bounded away from 0, \( \beta_t(1) \) evolves on a faster time scale than \( \beta_t(0) \). In calculating the limit value of (5.19), we first let \( \beta_t(1) \) reach its own “limit”, and then let \( \beta_t(0) \) go to its own limit point.

Given the assumption of Gaussian distributions,

\[ \ln \frac{A_t(0)}{A_t(1)} = -\frac{(p_t - \beta_t(0) z_t)^2}{2 \text{MSE}_t(0)} + \frac{(p_t - \beta_t(1) z_t)^2}{2 \text{MSE}_t(1)} + \frac{1}{2} \ln \left( \frac{\text{MSE}_t(1)}{\text{MSE}_t(0)} \right) \]  

(5.19)

Since the decision maker has rational expectations, he can calculate the mean squared error of each forecaster accurately. Thus, instead of \( \text{MSE}^*_t(0) \), the decision maker is using \( \text{MSE}_t(0) \). Note that

\[ \mathbb{E}\left( \frac{(p_t - \beta_t(0) z_t)^2}{2 \text{MSE}_t(0)} \right) = \mathbb{E}\left( \frac{(p_t - \beta_t(1) z_t)^2}{2 \text{MSE}_t(1)} \right) = 1 \]

since both are the normalized mean forecasting error.

Recall that

\[ \frac{1}{t} \log \frac{1 - \pi_t}{\pi_t} = \frac{1}{t} \sum_{k=1}^{t} \log \frac{A_k(0)}{A_k(1)}. \]

For a large \( t \), the right hand side is completely determined with probability 1 by

\[ \lim_{t \to \infty} \mathbb{E} \frac{1}{2} \ln \left( \frac{\text{MSE}_t(1)}{\text{MSE}_t(0)} \right) = \frac{1}{2} \ln \left( \frac{(1 - \alpha \rho \pi)^2 \Sigma \sigma_z^2 + \sigma^2}{(\alpha \rho \pi)^2 \Sigma^* (1) \sigma_z^2 + \sigma^2} \right) \]

which is positive if and only if

\[ \pi < \frac{1}{2 \alpha \rho}. \]

Now we prove that if

\[ \pi = \lim_{t \to \infty} \pi_t, \]

then \( \pi = 0 \) or 1 with probability 1. If \( \pi < \frac{1}{2 \alpha \rho} \), then

\[ \lim_{t \to \infty} \frac{1}{t} \log \frac{1 - \pi_t}{\pi_t} > 0 \]
with probability 1, which implies that \( \pi_t \to 0 \), contrary to our hypothesis that \( \pi > 0 \). Similarly, if \( \pi > \frac{1}{2\alpha \rho} \), then
\[
\lim_{t \to \infty} \frac{1}{t} \log \frac{1 - \pi_t}{\pi_t} < 0
\]
with probability 1, which implies that \( \pi_t \to 1 \), contrary to our hypothesis that \( \pi < 1 \).

5.4. **Self-confirming.** We have shown that \( \pi_t \to 0 \) or 1. It remains to show that \( \pi_t \) evolves at a slower time scale than \( \beta_t(0) \), in order to confirm \( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) forecasters’ belief that \( \pi_t \) evolves at a slower time scale that they can treat \( \pi_t \) as a known constant.

A simple calculation shows
\[
t(\pi_t - \pi_{t-1}) = \frac{t(e^{(t-1)\phi_{t-1}} - e^{t\phi_t})}{(1 + e^{t\phi_t})(1 + e^{(t-1)\phi_{t-1}})}.
\]
As \( t \to \infty \), we know \( \phi_t \to \phi^* \) with probability 1. We also know \( t(\phi_t - \phi_{t-1}) \) is uniformly bounded. Hence, we have
\[
\lim_{t \to \infty} t(\pi_t - \pi_{t-1}) = \lim_{t \to \infty} \frac{t \left( e^{-\phi^*} - 1 \right) e^{t\phi^*}}{(1 + e^{t\phi^*})(1 + e^{(t-1)\phi^*})} = (e^{-\phi^*} - 1) \lim_{t \to \infty} \frac{t}{(1 + e^{-t\phi^*})(1 + e^{t\phi^*} e^{-\phi^*})}
\]
Finally, notice that for both \( \phi^* > 0 \) and \( \phi^* < 0 \) the denominator converges to \( \infty \) faster than the numerator.

5.5. **Domain of Attraction.** It is helpful to figure out the domain of attraction of each locally stable point. Comparing the likelihoods, we can compute the domain of attraction for \( (\pi, \beta(0), \beta(1)) = (0, \frac{\delta}{1 - \rho \alpha}, \frac{\delta}{1 - \rho \alpha}) \) is the interior of
\[
\mathcal{D}_0 = \left\{ (\pi, \beta(0), \beta(1)) \mid E \log \frac{A_t(0)}{A_t(1)} \geq 0 \right\}
\]
which is roughly the area of
\[
\left\{ (\pi, \beta(0), \beta(1)) \mid \left( \beta(0) - \frac{\delta}{1 - \alpha \rho} \right)^2 < (1 - 2\alpha \rho \pi) \sigma^2 \left( \frac{1 - \alpha \rho \pi}{1 - \alpha \rho} \right)^2 \right\}
\]
where the mean forecasting error of \( \mathcal{M}_0 \) is smaller than that of \( \mathcal{M}_1 \). The difference is caused by the fact that the expected likelihood ratio differs from the expected mean forecasting error.

The interior of the complement of \( \mathcal{D}_0 \) is the domain of attraction for \( (\pi, \beta(0), \beta(1)) = (1, \frac{\delta}{1 - \rho \alpha}, \frac{\delta}{1 - \rho \alpha}) \)

For small \( \sigma_v > 0 \), one can imagine \( \mathcal{D}_0 \) as a narrow “cone” in the space of \( (\beta(0), \pi) \), with its apex at \( (\beta(0), \pi) = \left( \frac{\delta}{1 - \alpha \rho}, \frac{1}{2\alpha \rho} \right) \) and its base along the line \( \pi = 0 \), where \( \beta(0) \) is in \( \left[ \frac{\delta}{1 - \alpha \rho} - \frac{\sigma_v}{1 - \alpha \rho}, \frac{\delta}{1 - \alpha \rho} + \frac{\sigma_v}{1 - \alpha \rho} \right] \). Figure 11 plots \( \mathcal{D}_0 \) for the baseline parameter values used in the following simulations. The formal analysis will make the notion of being “narrow” precise.
6. Duration Time

We have three endogenous variables \((\pi_t, \beta_t(0), \beta_t(1))\), which converge to one of the two locally stable points: \((0, \delta/(1 - \alpha \rho), \delta/(1 - \alpha \rho))\) or \((1, \delta/(1 - \alpha \rho), \delta/(1 - \alpha \rho))\). Let us identify a specific stable point by the value of \(\pi_t\) at the stable point. Similarly, let \(D_0\) be the domain of attraction to \(\pi_t = 0\), and \(D_1\) be the domain of attraction to \(\pi_t = 1\).

For fixed \(\sigma^2_v > 0\), the distribution of \((\pi_t, \beta_t(0), \beta_t(1))\) assigns a large weight to either of the two locally stable points as \(t \to \infty\). Our main interest is the limit of this probability distribution as \(\sigma^2_v \to 0\).

To calculate the limit probability distribution, we need to calculate the probability that \((\pi_t, \beta_t(0), \beta_t(1))\) escapes from the domain of attraction of the locally stable point. The standard results from the large deviation theory say that the escape probability can be parameterized by the large deviation rate function.

**Lemma 6.1.** There exists \(r_0 \in [0, \infty]\) so that

\[
- \lim_{\sigma^2_v \to 0} \lim_{t \to \infty} \frac{1}{t} \log P \left( \exists t, \ (\pi_t, \beta_t(0), \beta_t(1)) \in D_0 \mid (\pi_1, \beta_1(0), \beta_1(1)) = \left(1, \frac{\delta}{1 - \alpha \rho}, \frac{\delta}{1 - \alpha \rho}\right) \right) = r_0
\]

and \(\exists r_1 \in [0, \infty]\) so that

\[
- \lim_{\sigma^2_v \to 0} \lim_{t \to \infty} \frac{1}{t} \log P \left( \exists t, \ (\pi_t, \beta_t(0), \beta_t(1)) \in D_1 \mid (\pi_1, \beta_1(0), \beta_1(1)) = \left(0, \frac{\delta}{1 - \alpha \rho}, \frac{\delta}{1 - \alpha \rho}\right) \right) = r_1.
\]

Large deviation parameter \(r_i\) \((i = 1, 2)\) quantifies how difficult it is to escape from \(D_i\), with \(r_i = \infty\) meaning that the escape never occurs, and \(r_i = 0\) meaning that the escape occurs with probability 1.
To calculate the relative duration times of \((\pi_t, \beta_t(0), \beta_t(1))\) around each locally attractive boundary point, we need to compute the following ratio

\[
\lim_{\sigma_v^2 \to 0} \lim_{t \to \infty} \frac{P\left( \exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in D_0 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(1, \frac{\delta}{1-\alpha \rho}, \frac{\delta}{1-\alpha \rho}\right) \right)}{P\left( \exists t, (\pi_t, \beta_t(0), \beta_t(1)) \in D_1 \mid (\pi_0, \beta_0(0), \beta_0(1)) = \left(0, \frac{\delta}{1-\alpha \rho}, \frac{\delta}{1-\alpha \rho}\right) \right)}.
\]

Note that \((\pi_t, \beta_t(0), \beta_t(1))\) stays in the neighborhood of \(1, \frac{\delta}{1-\alpha \rho}, \frac{\delta}{1-\alpha \rho}\) almost always in the limit, if \(r_0 < r_1\), and vice versa.

**Proposition 6.2.**

\(r_1 > r_0\).

*In the limit, the TVP model prevails, pushing out the constant parameter model.*

**Proof.** See Appendix B. \(\square\)

If \(\sigma_v^2 > 0\) is small, it is extremely difficult to detect whether \(\mathcal{M}_1\) is misspecified in the sense that the forecaster includes a variable \(\epsilon_{v,t}\) which does not exist in the rational expectations equilibrium. For fixed \(\sigma_v^2 > 0\), the asymptotic distribution of \(\pi_t\) assigns a large weight to 1 and 0, since 1 and 0 are locally stable points. Between the two locally stable points, we are interested in which locally stable point is more salient than the other. One way to determine which equilibrium is more salient than the other would be to compare the amount of time when \(\pi_t\) stays in a small neighborhood of each locally stable point. For fixed \(T\), \(\sigma_v^2\) and \(\varepsilon > 0\), define

\[ T_1 = \{ t \leq T \mid |\pi_t - 1| < \varepsilon \} \]

as the number of periods when \(\pi_t\) is within a small neighborhood of 1. Since 0 and 1 are only the two locally stable points, \(\pi_t\) stays in the neighborhood of 0 most of \(T - T_1\) periods, if not all.

As a corollary of Proposition 6.2, we can show that for a small \(\sigma_v^2 > 0\), \(\pi_t\) stays in the neighborhood of 1 almost always.

**Theorem 6.3.**

\[
\lim_{\sigma_v^2 \to 0} \lim_{T \to \infty} \frac{T_1}{T} = 1
\]

with probability 1.

The TVP model asymptotically dominates in the sense that the TVP model is used ‘almost always’. This is because it is better able to react to the volatility that it itself creates. Although \(\mathcal{M}_1\) is misspecified, this equilibrium must be learned via some adaptive process. What our result shows is that this learning process can be subverted by the mere presence of misspecified alternatives, even when the correctly specified model would converge if considered in isolation. This result therefore echoes the conclusions of Sargent (1993), who notes that adaptive learning models often need a lot of ‘prompting’ before they converge. Elimination of misspecified alternatives can be interpreted as a form of prompting.
7. Discussion

7.1. Relaxing dogmatic beliefs. Normally, with exogenous data, it would make no difference whether a parameter known to lie in some interval is estimated by mixing between the two extremes, or by estimating it directly. With endogenous data, however, this could make a difference. What if the agent convexified the model space by estimating \( \sigma_v^2 \) directly, via some sort of nonlinear adaptive filtering algorithm (e.g., Mehra (1972)), or perhaps by estimating a time-varying gain instead, via an adaptive step-size algorithm (Kushner and Yang (1995))? Although \( \pi = 1 \) is locally stable against nonlocal alternative models, would it also be stable against local alternatives?

In this case, there is no model averaging. There is just one model, with \( \sigma_v^2 \) viewed as an unknown parameter to be estimated. To address the stability question we exploit the connection between \( \sigma_v^2 \) and the steady-state gain, \( \gamma \). Because the data are endogenous, we must employ the macroeconomist’s ‘big \( K \), little \( k \)’ trick, which in our case we refer to as ‘big \( \Gamma \), little \( \gamma \)’. That is, our stability question can be posed as follows: Given that data are generated according to the aggregate gain parameter \( \Gamma \), would an individual agent have an incentive to use a different gain, \( \gamma \)? If not, then \( \gamma = \Gamma \) is a Nash equilibrium gain, and the associated \( \sigma_v^2 > 0 \) represents self-confirming parameter instability. The stability question can then be addressed by checking the (local) stability of the best response map, \( \gamma = B(\Gamma) \), at the self-confirming equilibrium.

To simplify the analysis, we consider a special case, where \( z_t = 1 \) (i.e., \( \rho = 1 \) and \( \sigma_z = 0 \)). The true model becomes

\[
p_t = \delta + \alpha E_t p_{t+1} + \sigma_t
\]

and the agent’s perceived model becomes

\[
p_t = \beta_t + \sigma_t
\]
\[
\beta_t = \beta_{t-1} + \sigma_v v_t
\]

where \( \sigma_v \) is now considered to be an unknown parameter. Note that if \( \sigma_v^2 > 0 \), the agent’s model is misspecified. As in Sargent (1999), the agent uses a random walk to approximate a constant mean. Equations (5.18)-(5.19) represent an example of Muth’s (1960) ‘random walk plus noise’ model, in which constant gain updating is optimal. To see this, write \( p_t \) as the following ARMA(1,1) process

\[
p_t = p_{t-1} + \varepsilon_t - (1 - \Gamma)\varepsilon_{t-1} \quad \Gamma = \frac{\sqrt{4s + s^2} - s}{2} \quad \sigma_v^2 = \frac{\sigma^2}{1 - \Gamma}
\]

where \( s = \sigma_v^2/\sigma^2 \) is the signal-to-noise ratio. Muth (1960) showed that optimal price forecasts, \( E_t p_{t+1} = \hat{p}_{t+1} \), evolve according to the constant gain algorithm

\[
\hat{p}_{t+1} = \hat{p}_t + \Gamma (p_t - \hat{p}_t)
\]

This implies that the optimal forecast of next period’s price is just a geometrically distributed average of current and past prices,

\[
\hat{p}_{t+1} = \left( \frac{\Gamma}{1 - (1 - \Gamma)\Gamma} \right) p_t
\]
Substituting this into the true model in (7.20) yields the actual price process as a function of aggregate beliefs

\[
p_t = \frac{\delta}{1 - \alpha} + \frac{1 - (1 - \Gamma)L}{1 - (1 - \alpha\Gamma)L} \epsilon_t
\]

\[
≡ \bar{p} + f(L; \Gamma)\tilde{\epsilon}_t
\]

Now for the ‘big \(\Gamma\), little \(\gamma\)’ trick. Suppose prices evolve according (7.26), and that an individual agent has the perceived model

\[
p_t = \frac{1 - (1 - \gamma)L}{1 - L}u_t
\]

\[
≡ h(L; \gamma)u_t
\]

What would be the agent’s optimal gain? The solution of this problem defines a best response map, \(\gamma = B(\Gamma)\), and a fixed point of this mapping, \(\gamma = B(\gamma)\), defines a Nash equilibrium gain. Note that the agent’s model is misspecified, since it omits the constant that appears in the actual prices process in (7.26). The agent needs to use \(\gamma\) to compromise between tracking the dynamics generated by \(\Gamma > 0\), and fitting the omitted constant, \(\bar{p}\). This compromise is optimally resolved by minimizing the Kullback-Leibler (KLIC) distance between equations (7.26) and (7.27)

\[
\gamma^* = B(\Gamma) = \arg\min_{\gamma} \left\{ E[h(L; \gamma)^{-1}(\bar{p} + f(L; \Gamma)\tilde{\epsilon}_t)]^2 \right\}
\]

\[
= \arg\min_{\gamma} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \log H(\omega; \gamma) + \sigma^2 H(\omega; \gamma)^{-1}F(\omega; \Gamma) + \bar{p}^2 H(0)^{-1} \right]d\omega \right\}
\]

where \(F(\omega) = f(e^{-i\omega})f(e^{i\omega})\) and \(H(\omega) = h(e^{-i\omega})h(e^{i\omega})\) are the spectral densities of \(f(L)\) in (7.20) and \(h(L)\) in (7.27). Although this problem cannot be solved with pencil and paper, it is easily solved numerically. Figure 8 plots the best response map using the same benchmark parameter values as before (except, of course, \(\rho = 1\) now)

Not surprisingly, the agent’s optimal gain increases when the external environment becomes more volatile, i.e., as \(\Gamma\) increases. What is more interesting is that the slope of the best response mapping is less than one. This means the equilibrium gain is stable. If agents believe that parameters are unstable, no single agent can do better by thinking they are less unstable. Figure 8 suggests that the best response map intersects the 45 degree line somewhere in the interval \((.10, .15)\). This suggests that the value of \(\sigma^2\) used for the benchmark TVP model in section 4 was a little too small, since it implied a steady-state gain of .072.

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14See Sargent (1999, chpt. 6) for another example of this problem.
15Note, the unit root in the perceived model in (5.24) implies that its spectral density is not well defined. (It is infinite at \(\omega = 0\)). In the numerical calculations, we approximate by setting \((1 - L) = (1 - \eta L)\), where \(\eta = .995\). This means that our frequency domain objective is ill-equipped to find the degenerate fixed point where \(\gamma = \Gamma = 0\). When this is the case, the true model exhibits i.i.d fluctuations around a mean of \(\delta/(1 - \alpha)\), while the agent’s perceived model exhibits i.i.d fluctuations around a mean of zero. The only difference between these two processes occurs at frequency zero, which is only being approximated here.
$\mathcal{M}_0$ forecaster can estimate $\Sigma^*(1)$ by

$$
\frac{1}{T} \sum_{t=1}^{T} (\beta(t) - \bar{\beta}(1))^2
$$

which is bounded away from 0, as long as $\sigma^2_\nu > 0$. An important observation is that the estimation of $\sigma^2_\nu$ or $\Sigma^*(1)$ does not affect the speed of evolution of $\beta(t)$ or $\beta(t)$. Thus, if each forecaster estimates $\sigma^2_\nu$ and $\Sigma^*(1)$ respectively, the hierarchy of the time scale among $\beta(t)$, $\beta(t)$ and $\pi_t$ continues to hold and we can follow exactly the same steps to analyze the dynamics of $(\beta(t), \beta(t), \pi_t)$.

7.2. Judgment. Our framework can be easily extended to cover Bullard, Evans, and Honkapohja (2008). Let us consider the baseline model

$$p_t = \delta z_t + \alpha E_t p_{t+1} + \sigma \epsilon_t$$

as before, where $z_t$ evolves according to AR(1) process:

$$z_t = \rho_z z_{t-1} + \epsilon_{z,t}$$

where $\rho_z \in (0,1)$ and $\epsilon_{z,t}$ is i.i.d. Gaussian white noise with variance $\sigma^2_z > 0$.

Let us now assume that the decision maker’s perceived model is

$$p_t = \beta_{t-1} z_t + \xi_t + \sigma \epsilon_t$$

$$\beta_t = \beta_{t-1} + \epsilon_{v,t}$$

(7.28)  (7.29)

where $\epsilon_t$ and $\epsilon_{v,t}$ are mutually independent i.i.d. white noise. The decision maker perceives

$$E \epsilon_{v,t}^2 = \sigma^2_v > 0.$$ 

(7.30)

Following Bullard, Evans, and Honkapohja (2008), we call $\xi_t$ judgment, which evolves according to AR(1) process:

$$\xi_t = \rho \xi_{t-1} + \epsilon_{\xi,t}$$

(7.31)
where $\rho_\xi \in (0, 1)$ and $\epsilon_{\xi,t}$ is an i.i.d. Gaussian white noise orthogonal to other variables. This assumption implies that the judgment $\xi_t$ at time $t$ influence the future evolution of $p_t$, but its impact is vanishing at a geometric rate. As long as the judgment variable is orthogonal to other variables, the judgment variable $\xi_t$ only increases the volatility of $p_t$.

Bullard, Evans, and Honkapohja (2008) assume the decision maker selects one model from a set of models, one of which includes the judgment variable, based upon the performance of different models. Let us instead assume that the decision maker averages forecasts across models, one without the judgment variable, where the corresponding perceived law of motion is $M_0$:

$$p_t = \beta_{t-1} z_t + \sigma \epsilon_t$$  \hspace{1cm} (7.31)

$$\beta_t = \beta_{t-1} + \epsilon_{v,t}$$  \hspace{1cm} (7.32)

where $\sigma_v = 0$, and another model that is based upon (7.28) and (7.29), which we call $M_2$. Without a judgment variable, the decision maker's perceived law of motion is precisely the correctly specified model. Thus, it is natural to assume $\epsilon_{v,t} = 0 \forall t \geq 0$ so that $\forall t \geq 1$, $\beta_t = \beta$ for some constant $\beta$.

We can invoke exactly the same analysis to the case of model averaging over $M_0$ and $M_2$. Let $\pi_t$ be the weight assigned to $M_2$.

**Proposition 7.1.** If $\alpha \rho_z > 0$ is sufficiently close to 1, then the dynamics of $\pi_t$ has two stable stationary points: 1 and 0. As $\sigma_v \to 0$, the proportion of time that $\pi_t$ stays in the neighborhood of 1 converges to 1.

Note that any consistent estimator of $\sigma_v$ is bounded away from 0, as long as $z_t$ and $\xi_t$ have non-degenerate correlation. Also, when the decision maker switches away from $M_0$ to $M_2$, the switching occurs very quickly so that his behavior looks very much like a selection of $M_2$ over $M_0$.

### 8. Conclusion

Parameter instability is a fact of life for applied econometricians. This paper has proposed one explanation for why this might be. We show that if econometric models are used in a less than fully understood self-referential environment, parameter instability can become a self-confirming equilibrium. Parameter estimates are unstable simply because model-builders think they might be unstable.

Clearly, this sort of volatility trap is an undesirable state of affairs, which raises questions about how it could be avoided. There are two main possibilities. First, not surprisingly, better theory would produce better outcomes. The agents here suffer bad outcomes because they do not fully understand their environment. If they knew the true model in (2.1), they would know that data are endogenous, and would avoid reacting to their own shadows. They would simply estimate a constant parameters reduced form model. A second, and arguably more realistic possibility, is to devise econometric procedures that are more robust to misspecified endogeneity. In Cho and Kasa (2015), we argue that in this sort of environment, model selection might actually be preferable to model averaging.

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16Bullard, Evans, and Honkapohja (2010) investigated the case where the judgment variable is correlated with fundamentals, and show the existence of an equilibrium with judgment.
If agents selected either a constant or TVP model based on sequential application of a specification or hypothesis test, the constant parameter model would prevail, as it would no longer have to compete with the TVP model.
Appendix A. Proof of Proposition 4.2

This result is quite intuitive. With two full support probability distributions, you can never conclude that a history of any finite length couldn’t have come from either of the distributions. Slightly more formally, if the distributions have full support, they are mutually absolutely continuous, so the likelihood ratio in (4.16) is strictly bounded between 0 and some upper bound \( B \). To see why \( \pi_t < 1 \) for all \( t \), notice that \( \pi_{t+1} \leq \pi_t + \pi_t(1-\pi_t)M \) for some \( M < 1 \), since the likelihood ratio is bounded by \( B \). Therefore, since \( \pi + \pi(1-\pi) \in [0,1] \) for \( \pi \in [0,1] \), we have

\[
\pi_{t+1} \leq \pi_t + \pi_t(1-\pi_t)M
\]

\[
< \pi_t + \pi_t(1-\pi_t)
\]

\[
\leq 1
\]

and so the result follows by induction. The argument for why \( \pi_t > 0 \) is completely symmetric.

Appendix B. Proof of Proposition 6.2

B.1. Preliminaries. \( \beta_t(1) \) moves at the fastest time scale, followed by \( \beta_t(0) \) and then \( \pi_t \). The same reasoning also shows the domain of attraction for \( \pi = 0 \) is

\[
D_0 = \left\{ (\pi, \beta(0), \beta(1)) \mid \mathbb{E} \log \frac{A_t(0)}{A_t(1)} > 0 \right\}
\]

and the domain of attraction for \( \pi = 1 \) is

\[
D_1 = \left\{ (\pi, \beta(0), \beta(1)) \mid \mathbb{E} \log \frac{A_t(0)}{A_t(1)} < 0 \right\}.
\]

Since \( \beta_t(1) \) does not trigger the escape from one domain of attraction to another, let us focus on \( (\pi, \beta(0)) \), assuming that we are moving according to the time scale of \( \beta_t(0) \). A simple calculation shows that \( D_0 \) has a narrow symmetric shape of \( (\pi, \beta(0)) \), centered around

\[
\beta(0) = \frac{\delta}{1 - \alpha \rho}
\]

with the base

\[
\left( \frac{\delta}{1 - \alpha \rho} - d, \frac{\delta}{1 - \alpha \rho} + d \right)
\]

along the line \( \pi = 0 \) where

\[
d = \sqrt{\frac{\Sigma}{1 - \alpha \rho}}. \tag{B.33}
\]

Note that since \( \Sigma \to 0 \) as \( \sigma_v \to 0 \),

\[
\lim_{\sigma_v \to 0} d = 0.
\]

Define

\[
\bar{\pi} = \sup\{\pi \mid (\pi, \beta(0), \beta(1)) \in D_0\}
\]

which is \( 1/(2\alpha \rho) \).

Recall that

\[
\phi_t = \frac{1}{T} \sum_{k=1}^{t} \log \frac{A_k(0)}{A_k(1)}.
\]

Note that since \( \beta_t(0), \beta_t(1) \to \frac{\delta}{1 - \alpha \rho} \),

\[
\phi^* = \mathbb{E} \log \frac{A_0(0)}{A_0(1)} = \mathbb{E} \frac{1}{2} \log \frac{\text{MSE}(1)}{\text{MSE}(0)}
\]

is defined for \( \beta_t(0) = \beta_t(1) = \frac{\delta}{1 - \alpha \rho} \), and \( \pi = 1 \) or 0.
We know that $\pi = 1$ and $\pi = 0$ are only limit points of $\{\pi_t\}$. Define $\phi^*$ as $\phi^*$ evaluated at $(\beta(1), \beta(0), \pi) = (\delta \frac{1}{1-\rho}, \frac{\delta}{1-\rho}, 1)$ and similarly, $\phi^*_+$ as $\phi^*$ evaluated at $(\beta(1), \beta(0), \pi) = (\delta \frac{1}{1-\rho}, \frac{\delta}{1-\rho}, 0)$. A straightforward calculation shows

$$\phi^-_* < 0 < \phi^*_+$$

and

$$\phi^*-\phi^*_+ > 0.$$

Recall $r_0$ and $r_1$ are the rate functions of $D_0$ and $D_1$. For fixed $\sigma_v > 0$, define

$$r_0(\sigma_v) = - \lim_{t \to \infty} \frac{1}{t} \log P \left( \exists t, (\beta_t(1), \beta_t(0), \pi_t) \in D_0 \mid (\beta_1(1), \beta_1(0), \pi_1) = \left(\frac{\delta}{1-\rho}, \frac{\delta}{1-\rho}, 1\right) \right)$$

and

$$r_1(\sigma_v) = - \lim_{t \to \infty} \frac{1}{t} \log P \left( \exists t, (\beta_t(1), \beta_t(0), \pi_t) \in D_1 \mid (\beta_1(1), \beta_1(0), \pi_1) = \left(\frac{\delta}{1-\rho}, \frac{\delta}{1-\rho}, 0\right) \right)$$

Then,

$$r_0 = \lim_{\sigma_v \to 0} r_0(\sigma_v) \quad \text{and} \quad r_1 = \lim_{\sigma_v \to 0} r_1(\sigma_v).$$

Our goal is to show that $\exists \sigma_v > 0$ such that

$$\inf_{\sigma_v \in (0, \sigma_v)} r_1(\sigma_v) - r_0(\sigma_v) > 0.$$

**B.2. Escape probability from $D_1$.** Consider a subset of $D_1$

$$D'_1 = \{(\beta(1), \beta(0), \pi) \mid \pi > \bar{\pi}\}.$$ 

For fixed $\sigma_v > 0$, define

$$r_1^*(\sigma_v) = - \lim_{t \to \infty} \frac{1}{t} \log P \left( \exists t, (\beta_t(1), \beta_t(0), \pi_t) \notin D'_1 \mid (\beta_1(1), \beta_1(0), \pi_1) = \left(\frac{\delta}{1-\rho}, \frac{\delta}{1-\rho}, 1\right) \right)$$

and

$$r_1^* = \lim_{\sigma_v \to 0} r_1^*(\sigma_v).$$

Note that

$$\exists t, (\beta_t(1), \beta_t(0), \pi_t) \notin D'_1$$

if and only if

$$\pi_t < \bar{\pi}$$

and

$$\phi_t > 0.$$

We know

$$\lim_{t \to \infty} \log P \left( \exists t, \phi_t > 0 \mid \phi_1 = \phi^*_+ \right) = r_1^*(\sigma_v).$$

We claim that

$$r_1^* > 0. \quad (B.34)$$

The substance of this claim is that $r_1^*$ cannot be equal to 0. This statement would have been trivial, if $\phi^*_+$ is uniformly bounded away from 0. In our case, however,

$$\lim_{\sigma_v \to 0} \phi^*_+ = 0$$

which implies $\sum \to 0$. Note that

$$\phi_t > 0$$

if and only if

$$\phi_t - \phi^*_+ > -\phi^*_+$$

if and only if

$$\frac{1}{t} \sum_{k=1}^{t} \left[ \log \frac{A_t(0)}{A_t(1)} = \mathbb{E} \log \frac{A_t(0)}{A_t(1)} \right] > -\phi^*_+.$$
A straightforward calculation shows

\[
\frac{1}{t} \sum_{k=1}^{t} \left[ \log \frac{A_t(0)}{A_t(1)} - \mathbb{E} \log \frac{A_t(0)}{A_t(1)} \right] > -\frac{\phi^*}{\Sigma}.
\]  

(B.35)

A straightforward calculation shows

\[
\lim_{\sigma \to 0} -\frac{\phi^*}{\Sigma} = \frac{\sigma^2}{\sigma^2} \left( \alpha \rho - \frac{1}{2} \right) > 0.
\]

It is tempting to conclude that we can invoke the law of large numbers to conclude that the sample average has a finite but strictly positive rate function. However,

\[
\frac{\log A_t(0)}{\lambda A_t(1)} - \mathbb{E} \log \frac{A_t(0)}{A_t(1)}
\]

is not a martingale difference. Although its mean converges to 0, we cannot invoke Cramér’s theorem to show the existence of a positive rate function. Instead, we shall invoke Gärtner Ellis theorem (Dembo and Zeitouni (1998)).

We claim that

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} e^{\lambda Y_t} = 0.
\]

A simple calculation shows

\[
\mathbb{E} \log \frac{A_t(0)}{A_t(1)} - \mathbb{E} \log \frac{A_t(0)}{A_t(1)} = \frac{1}{2} \log \frac{\Sigma \tau(1) \sigma_{z,t} + \sigma^2}{\Sigma \sigma_z^2 + \sigma}.
\]

Since \( \Sigma \tau(1) \to \Sigma > 0 \), and \( \Sigma \tau(1) \) is bounded, \( \exists M > 0 \) such that

\[
\Sigma \tau(1) \leq M
\]

and \( \forall \epsilon > 0, \exists T(\epsilon) \) such that \( \forall t \geq T(\epsilon) \),

\[
\left| \mathbb{E} \log \frac{A_t(0)}{A_t(1)} - \mathbb{E} \log \frac{A_t(0)}{A_t(1)} \right| \leq \epsilon.
\]

Thus, as \( t \to \infty \),

\[
\frac{1}{t} \log \mathbb{E} e^{\lambda Y_t} \leq \frac{1}{t} \log \mathbb{E} e^{\lambda |\epsilon|} + \frac{2T(\epsilon)M}{t} = |\lambda| \epsilon + \frac{2T(\epsilon)M}{t} \to |\lambda| \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we have the desired conclusion.

We conclude that the \( H \) functional (a.k.a., the logarithmic moment generating function) of

\[
\frac{1}{t} \sum_{k=1}^{t} \left[ \log \frac{A_t(0)}{A_t(1)} - \mathbb{E} \log \frac{A_t(0)}{A_t(1)} \right]
\]

is precisely

\[
H(\lambda) = \lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} e^{\lambda Y_t}.
\]

That means, the large deviation properties of the left hand side of (B.35) is the same as the large deviation properties of \( Y_t \). Since \( Y_t \) is the sample average of the martingale difference, the standard argument of the
large deviation theory implies that its rate function is strictly positive for given $\sigma_v > 0$. We normalized the martingale difference by dividing each term by $\sum$ so that the second moment of
\[
\sum \log \frac{\Delta r(t)}{\Delta r(i)} - E \log \frac{\Delta r(t)}{\Delta r(i)}
\]
is uniformly bounded away from 0, even in the limit as $\sigma_v \to 0$. Hence,
\[
\lim_{\sigma_v \to 0} H(\lambda)
\]
does not vanish to 0, which could have happened if the second moment of the marginal difference converges to 0. By applying Gärtner Ellis Theorem, we conclude that $\exists r_1^*(\sigma_v) > 0$ such that
\[
\lim_{t \to \infty} \log P \left( \frac{1}{t} \sum_{k=1}^{t} \left[ \log \frac{\Delta r(t)}{\Delta r(i)} - E \log \frac{\Delta r(t)}{\Delta r(i)} \right] \geq -\frac{\phi^*}{\Sigma} \right) = \lim_{t \to \infty} \log P \left( Z_t \geq -\frac{\phi^*}{\Sigma} \right) = r_1^*(\sigma_v)
\]
and
\[
\lim \inf_{\sigma_v \to 0} r_1^*(\sigma_v) = r_1^* > 0
\]
as desired.

**B.3. Escape probability from $D_0$.** Recall that $\beta_t(0)$ evolves according to
\[
\beta_{t+1}(0) = \beta_t(0) + \frac{\sum_t(0)z_t^2}{\sigma^2 + \sum_t(0)z_t^2} [p_t - \beta_t(0)z_t].
\]
At $\pi_t = 0$, the forecasting error is
\[
p_t = \beta_t(0)z_t = (1 - \alpha \rho) \left[ \frac{\delta}{1 - \alpha \rho} - \beta_t(0) \right] z_t + \sigma \epsilon_t.
\]
Note that the forecasting error is independent of $\sigma_v$. Following Dupuis and Kushner (1989), we can show that $\forall \sigma_v > 0$, $\exists r_0^*(\sigma_v) > 0$ such that
\[
\lim_{t \to \infty} \frac{1}{t} \log P \left( \left| \beta_t(0) - \frac{\delta}{1 - \alpha \rho} \right| > d \mid \beta_t(0) = \frac{\delta}{1 - \alpha \rho} \right) = r_0^*(d)
\]
and
\[
\lim_{d \to 0} r_0^*(d) = 0.
\]

**B.4. Conclusion.** Recall [B.33] to notice that
\[
\lim_{\sigma_v \to 0} d = 0.
\]
Thus, we can find $\sigma_v > 0$ such that $\forall \sigma_v \in (0, \sigma_v)$,
\[
r_0^*(d) < \frac{r_1^*}{2} = \frac{1}{2} \lim \inf_{\sigma_v \to 0} r_1^*(\sigma_v).
\]
Observe
\[
r_0(\sigma_v) \leq r_0^*(d)
\]
since the exit occurs at the most likely exit point at which $r_0$ is determined, while $r_0^*(d)$ is determined at a particular exit point. Since $D_1 \subset D_1$
\[
r_1^*(\sigma_v) \leq r_1(\sigma_v).
\]
Thus, for any $\sigma_v > 0$ sufficiently close to 0,
\[
r_0(\sigma_v) \leq r_0^*(d) < \frac{r_1^*}{2} \leq r_1^*(\sigma_v) \leq r_1(\sigma_v).
\]
from which
\[
\inf_{\sigma_v \in (0, \sigma_v)} r_1(\sigma_v) - r_0^*(\sigma_v) > 0
\]
follows.
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REFERENCES


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