Business Networks, Production Chains, and Productivity:
A Theory of Input-Output Architecture*

Ezra Oberfield
Princeton University
ezraoberfield@gmail.com

October 20, 2013

Abstract

This paper develops a theory in which the network structure of production—who buys inputs from whom—is the endogenous outcome of individual choices, and studies how these choices shape productivity and the organization of production. Entrepreneurs produce using labor and exactly one intermediate input; the key decision is which other entrepreneur’s good to use as an input. Their choices collectively determine the economy’s equilibrium input-output structure. Despite the network structure, the model is analytically tractable, allowing for sharp characterizations of productivity and various micro-level characteristics. When the share of intermediate goods relative to labor in production is high, star suppliers emerge endogenously. This raises aggregate productivity as, in equilibrium, more supply chains are routed through higher-productivity techniques. As new techniques are discovered, entrepreneurs substitute across suppliers in response to changing input prices. Larger firms experience smaller reductions in cost but are important in propagating cost savings through the network.

Keywords: Networks, Productivity, Supply Chains, Ideas, Diffusion

JEL Codes: O31, O33, O47

*I am grateful for comments from Amanda Agan, Fernando Alvarez, Enghin Atalay, Gadi Barlevy, Marco Bassetto, Jarda Borovicka, Paco Buera, Jeff Campbell, Vasco Carvalho, Thomas Chaney, Meredith Crowley, Cristina De Nardi, Aspen Gorry, Joe Kaboski, Sam Kortum, Alejandro Justiniano, Robert Lucas, Devesh Raval, Rob Shimer, Nancy Stokey, Nico Trachter, Mark Wright, Andy Zuppann, and various seminar participants. I also thank Todd Messer and Matt Olson for excellent research assistance. All mistakes are my own.
Individual producers buy inputs from others and may sell their output for intermediate use, forming a network of production. A growing literature recognizes that aggregate productivity reflects both individual productivity and the economy’s network structure. These linkages determine how a change in one producer’s cost of production impacts other producers.¹

The formation of this network is the focus of this paper. I develop a theory in which the economy’s network structure reflects the decisions and interactions of individual producers. I then use the theory to address two questions: How do incentives facing individual producers shape the organization of production? And what is the impact of their choices on productivity, both for individual producers and for the economy as a whole?

The theory is based on the premise that there may be multiple ways to produce a good, each with a different set of inputs. In the model, each entrepreneur sells a particular good and searches for techniques to produce that good. Production requires labor and exactly one intermediate input. The key decision is which other entrepreneur’s good to use as an input. Each entrepreneur occasionally discovers a new technique to produce her good that uses another entrepreneur’s good as an input, with productivity specific to that input. The cost-effectiveness of a technique depends both on its productivity and on the cost of the associated input; when producing, each entrepreneur selects from the techniques she has discovered the one that delivers the lowest cost combination. These choices collectively determine the equilibrium input-output architecture of the economy.

In the model, aggregate production technology consists of an evolving set of relationships among producers.² The collection of known production techniques form a dynamic network comprising each entrepreneur’s potential suppliers and potential customers – others who

¹The literature on aggregate fluctuations has focused on how sectoral shocks propagate through an economy – see Long and Plosser (1983), Horvath (1998), Dupor (1999), Acemoglu et al. (2012), Foerster et al. (2011), and Atalay (2013). In the growth literature, Jones (2011, 2013) argues that misallocation in one sector raises input prices in other sectors, and the magnitude of the overall effect depends on the input-output structure. In the trade literature, Caliendo and Parro (2011) argue that the welfare impact of changes in tariffs in particular industries depends on the input-output structure.

²The theory emphasizes technological interdependence in the spirit of Rosenberg (1979): “The social payoff of an innovation can rarely be identified in isolation. The growing productivity of industrial economies is the complex outcome of large numbers of interlocking, mutually reinforcing technologies, the individual components of which are of very limited economic consequence by themselves. The smallest relevant unit of observation is seldom a single innovation but, more typically, an interrelated clustering of innovations.” (p. 28-29)
might use the entrepreneur’s good as an input. These jointly determine each firm’s size and its contribution to aggregate productivity. Innovation is the arrival of a new technique. Such an innovation represents both a new way to produce one good and a new use for a different good.\textsuperscript{3} Techniques that are not currently cost-effective still have value as the cost of the input may fall in the future. As new techniques are discovered, prices adjust, entrepreneurs substitute across suppliers, and cost savings diffuse through the network. How large those effects are hinge on the structure of the network.

I first characterize the model’s implications for cross-sectional patterns: how the techniques available to the various entrepreneurs and the choices they make shape the equilibrium network structure. A key result is about the emergence of star suppliers. These are individual entrepreneurs that, in equilibrium, sell their goods for intermediate use to many other entrepreneurs.\textsuperscript{4} For an entrepreneur to be a star supplier, there must be many other entrepreneurs that have techniques that would use her good as an input, and a large fraction of those potential customers must choose to use her good, which will depend on the price she charges. It turns out that even when there is little variation in prices that entrepreneurs charge, there can still be large differences in size; the feature of the environment that determines the prevalence of star suppliers is the share of intermediate goods in production relative to labor. Why does the intermediate goods share play the key role? Recall that when an entrepreneur selects a supplier, she considers both the match-specific productivity and the cost of the associated input of each of her techniques. When the share of intermediate goods in production is small, the cost of the inputs is less important as less is spent on purchasing inputs. Thus her choice of supplier will be less driven by her potential suppliers’ costs of production. Conversely, when the intermediate goods share is large, entrepreneurs with low production costs are selected as suppliers more systematically, and are therefore more likely to be star suppliers.

The emergence of star suppliers affects aggregate productivity in the following sense:

\textsuperscript{3}The idea that innovation is finding a new use for an existing good is related to recombinant growth of Weitzman (1998), in which innovation is finding a better way to combine inputs. Scherer (1982) and Pavitt (1984) show that roughly three quarters of innovations are for use by others outside the innovating firm’s sector.

\textsuperscript{4}Acemoglu et al. (2012) show that the prevalence of star suppliers is one factor that determines whether idiosyncratic shocks are relevant for aggregate fluctuations.
entrepreneurs’ selections jointly determine the supply chains used to produce each good. Aggregate productivity depends on the match-specific productivity of the technique used at each step in each of those supply chains. With a larger intermediate goods share, in equilibrium more of these supply chains get routed through the most productive techniques in the economy, raising aggregate productivity.

I next study how the network structure evolves over time and how the existing input-output architecture determines the impact of new techniques. The model predicts that entrepreneurs with low costs of production tend to remain with their suppliers longer than those with high costs, as the former are less likely to discover techniques more cost-effective than the ones they are already using. In addition, entrepreneurs that switch suppliers grow more than those that keep the same supplier; in the model, an entrepreneur would switch suppliers only if the new supplier delivered a lower cost, which would enable her to attract more customers.

These are informative about the larger question of how cost savings from new techniques diffuse through the network. An entrepreneur with a low production cost is unlikely to find a new lower-cost way to produce, and if she does, the cost reduction is likely to be small. However, her cost will fall when her supplier finds a new, more cost-effective technique. Because she is likely to have many customers (due to her already-low production cost), she can pass on those cost savings to many other entrepreneurs. Thus while star suppliers are unlikely to be sources of cost savings, they play an important role in propagating cost savings through the economy.

The implications of the model relate to two sets of papers in the network literature. The first set of papers, described in footnote (1), study sectoral input-output tables and analyze how changes in cost in particular sectors impact aggregate output. The second set of papers study models in which links form randomly between nodes. These specify a stochastic process for the arrival of links and study the resulting link patterns.\(^5\)

In contrast, here the network structure is endogenous in that it depends on prices in the natural way: when an entrepreneur charges a lower price, other entrepreneurs are more likely

\(^5\)Variants of the preferential attachment model of Barabasi and Albert (1999) have been used to explain the cross-sectional distribution of the number of suppliers (Atalay et al. (2011)) and the network structure of international shipments of goods (Chaney (2011)). See also Kelly et al. (2013).
to use her good as an input. Because the equilibrium network reflects individual choices, it features a rich set of implications for both the cross-sectional patterns and producer behavior over time. Further, these patterns determine aggregate productivity and influence how cost savings diffuse through the economy.

This paper thus makes a technical contribution in providing an approach to modeling the formation of a network. The model admits an analytical solution, which facilitates characterizing these patterns and how they are shaped by the environment. The endogeneity of the network also opens the door to analyzing how changes in policy or the environment would change incentives and thus the link formation process, or how changes in the set of potential suppliers or customers would impact productivity.

Finally, the model is written in such a way that it nests a simple version of Kortum (1997)\(^6\) when the share of intermediate goods in production is zero—a special case in which the network structure plays no role. This special case provides a backdrop against which one can see the network structure’s role in shaping economic outcomes.

The paper proceeds as follows: Section 1 describes a static version of the environment, setting up and solving a planner’s problem. Section 2 defines an equilibrium for a particular market structure and shows that there are equilibria that decentralize the planner’s problem. Section 3 describes cross-sectional features of the environment and describes how these relate to aggregate output. Section 4 extends the model to a dynamic setting, focusing on turnover in vertical relationships and the diffusion of cost-savings through the network. Section 5 concludes.

1 A Static Model

There is a unit mass of infinitely-lived entrepreneurs, each associated with producing a particular good.\(^7\) Each good is used for final consumption and potentially as an intermediate input by other firms. A representative consumer has Dixit-Stiglitz preferences over the goods with elasticity of substitution across varieties \(\varepsilon\), and supplies \(L\) units of labor inelastically.

\(^6\)The model is also related to Alvarez et al. (2008) and Lucas (2009).

\(^7\)The words entrepreneur and firm will be used interchangeably.
(a) The Set of Techniques

(b) Techniques that are Used

Figure 1: A Graphical Representation of the Input-Output Structure

Figure 1a gives an example of a set of techniques, \( \Phi \). Each node is an entrepreneur and edges correspond to techniques. An edge’s direction indicates which entrepreneur produces the output and which provides the input. An edge’s number represents the productivity of the technique. In Figure 1b, solid arrows are techniques that are used.

To produce, an entrepreneur must use a technique. For an entrepreneur, a technique is a method of producing her good using labor and exactly one other entrepreneur’s good as an intermediate input. Each technique is a production function fully described by the triple \( \phi = \{ \text{buyer}, \text{supplier}, \text{productivity} \} \). These are: (i) the good that is produced, \( b(\phi) \); (ii) the good used as an input, \( s(\phi) \); and (iii) a productivity parameter, \( z(\phi) \):

\[
y_b = \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} z(x_s l^{1-\alpha})
\]

where \( y_b \) is the quantity of output of good \( b(\phi) \), \( x_s \) is the quantity of good \( s(\phi) \) used as an input, and \( l \) is labor. Given a menu of techniques and input prices, each entrepreneur produces using the technique that delivers the best combination of input cost and productivity.

The set of all the techniques available to the different entrepreneurs, \( \Phi \), is a complete description of the production possibilities in the economy. Techniques arrive randomly via a process described below. Any realization of the set of techniques, \( \Phi \), has a representation as a weighted, directed graph. Figure 1a gives an example of a realization for an economy with a finite number of entrepreneurs. In the figure, entrepreneurs are represented by nodes. Each technique is a link connecting two nodes that has a direction corresponding to the flow of goods, indicating which entrepreneur would supply the intermediate input and which
produces the output. In addition, each edge has a number corresponding to the productivity of the technique. Some entrepreneurs have multiple ways to produce. For example, E has two ways of producing, one that uses good D and one that uses good B. In equilibrium E will select a single technique with which to produce. It is also possible for an entrepreneur to have no techniques; given the realization of \( \Phi \), it is infeasible for G to produce. To summarize, Figure 1a shows a complete description of the production possibilities in the economy. Figure 1b gives an example of entrepreneurs’ selections of which techniques to use (which will ultimately depend on prices). These selections correspond to vertical relationships we would observe in equilibrium and jointly determine the supply chains that are used to produce each good.

The set of techniques is a random set. The number of techniques an entrepreneur has to produce her good follows a Poisson distribution with mean \( M \). I will refer to \( M \) as the density of techniques in the network. For each of those techniques, the identity of the supplier \( s(\phi) \) is random and uniformly distributed across all firms in the economy. A technique’s productivity, \( z(\phi) \), is drawn from a fixed distribution with CDF \( H \). It is assumed that the support of \( H \) is bounded below by some \( z_0 > 0 \) and that the tail of the distribution is not too thick: there exists a \( \beta > \varepsilon - 1 \) such that \( \lim_{z \to \infty} z^\beta [1 - H(z)] = 0 \), a sufficient condition for aggregate output to be finite with probability one.

\( \Phi \) determines two sets of techniques that are directly relevant for each firm. Let \( U_j \equiv \{ \phi \in \Phi : b(\phi) = j \} \) be the techniques for which \( j \) is the buyer (those upstream from \( j \)) and let \( D_j \equiv \{ \phi \in \Phi : s(\phi) = j \} \) be the techniques for which \( j \) is the supplier (those downstream from \( j \)). These respectively determine \( j \)'s potential suppliers and potential customers.

\( \Phi \) also determines the supply chains available to each entrepreneur. A viable supply chain for entrepreneur \( j \) is a (possibly repeating) infinite sequence of techniques \( \{ \phi_k \}_{k=0}^{\infty} \) with the property that \( j = b(\phi_0) \) and \( s(\phi_k) = b(\phi_{k+1}) \) for each \( k \). It is feasible for an entrepreneur to produce only if she has at least one viable supply chain. The assumptions that production of each good requires the use of some other good as an input and that chains of these

---

8To keep the notation manageable, I abstract from any ex-ante differences in goods’ suitability for use as an intermediate input for any other type of good or for consumption. In many applications, ex-ante asymmetries across goods would be important (reflecting different industries or countries). The model can easily accommodate such asymmetries without loss of tractability; see working paper version.
relationships may continue indefinitely follow in the tradition of Leontief (1951) and other models with roundabout production.⁹

1.1 A Planner’s Problem

This section focuses on a planner’s problem in order to build intuition about the economic environment. Section 2 discusses a market structure with bilateral two-part pricing for each technique and shows there are equilibria that decentralize the planner’s solution.

Consider the problem of a planner that, taking the set of techniques Φ as given, makes production decisions and allocates labor to maximize the utility of the representative consumer. For each producer \( j \in [0, 1] \), the planner chooses the quantity to be produced for final consumption, \( y^0_j \). In addition, for each of \( j \)’s upstream techniques \( \phi \in U_j \), the planner chooses a quantity of labor, \( l(\phi) \), and a quantity of good \( s(\phi) \) for \( j \) to use as an intermediate input, \( x(\phi) \). It is feasible for each firm to produce using more than one technique. However, because each technique exhibits constant returns to scale, the use of a single technique will generically be optimal.

Formally, the planner faces the following problem:

\[
\max_{\{y^0_j, \{x(\phi), l(\phi)\}_{\phi \in U_j}\}_{j \in [0, 1]}} \left( \int_0^1 (y^0_j)^{\varepsilon-1} \, dj \right)^{\frac{\varepsilon}{\varepsilon-1}}
\]

subject to technological constraints:

\[
y^0_j + \sum_{\phi \in D_j} x(\phi) \leq \sum_{\phi \in U_j} \frac{1}{\alpha^\alpha (1 - \alpha)^{1-\alpha}} z(\phi) x(\phi)^\alpha l(\phi)^{1-\alpha}, \quad \forall j \in [0, 1]
\]

and the labor resource constraint:

\[
\int_0^1 \left( \sum_{\phi \in U_j} l(\phi) \right) \, dj \leq L
\]

The technological constraint for good \( j \) says that total usage of good \( j \)–output for consump-

⁹In the example with a finite set of firms, any viable supply chain must contain a cycle, just as two sectors might each supply inputs to the other in Leontief (1951). With a continuum of firms, a cycle is not necessary.
tion and for use as an input in other firms’ production—cannot exceed total production of good $j$.

Let $\lambda_j$ be the multiplier on the technological constraint for $j$, the marginal cost to the planner of good $j$. Let $w$ be the multiplier on the labor clearing constraint, the marginal cost to the planner of labor. Finally, let $Y^0 \equiv \left( \int_0^1 (y_j^0)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}}$ denote the aggregate of final output consumed by the representative consumer. The optimal choice of output for consumption of $j$ implies

$$\left( \frac{y_j^0}{Y^0} \right)^{-1/\varepsilon} = \lambda_j$$

(1)

For each technique $\phi \in \Phi$, the first order conditions with respect to $x(\phi)$ and $l(\phi)$ imply

$$\lambda_{b(\phi)} \frac{y(\phi)}{x(\phi)} \leq \lambda_{s(\phi)} \text{, with equality if } x(\phi) > 0$$

$$\lambda_{b(\phi)} (1 - \alpha) \frac{y(\phi)}{l(\phi)} \leq w \text{, with equality if } l(\phi) > 0$$

Together with the production function associated with $\phi$, these imply that $\lambda_{b(\phi)} \leq \frac{1}{z(\phi)} \lambda_{s(\phi)}^\alpha w^{1-\alpha}$, with equality if $y(\phi) > 0$. Collecting these conditions for each of $j$’s upstream techniques gives

$$\lambda_j = \min_{\phi \in U_j} \frac{1}{z(\phi)} \lambda_{s(\phi)}^\alpha w^{1-\alpha}$$

(2)

It will be convenient to define $q_j \equiv \frac{w}{\lambda_j}$ as a measure of the efficiency with which the planner can produce good $j$ in units of labor. Equation (2) can be written as

$$q_j = \max_{\phi \in U_j} z(\phi) q_{s(\phi)}^\alpha$$

(3)

with $q_j = 0$ if $U_j$ is empty. The efficiency delivered by each of $j$’s upstream techniques depends on the productivity of the technique and the efficiency with which the associated input can be produced. The efficiency with which the planner can produce $j$ is then the maximum delivered by any of the available techniques.

Aggregate output depends on firms’ efficiencies. Let $Q \equiv \left( \int_0^1 q_j^{\varepsilon-1} dj \right)^{\frac{1}{\varepsilon}}$. Lemma 1 shows that $Q$ measures aggregate productivity.

**Lemma 1.** In the optimal allocation, $Y^0 = QL$. 

8
Proof. In characterizing the allocation of resources, it will be useful to organize production into supply chains. Let $\mathcal{C}_j^\infty$ be the set of viable supply chains to produce $j$. For any chain $c \in \mathcal{C}_j^\infty$, we consider all of the resources used throughout the supply chain to produce final output of good $j$. Let $y_j^0(c)$ be the total amount of final output of $j$ produced using supply chain $c$. Let $l_{j,0}(c)$ and $x_{j,0}(c)$ be the labor and inputs used by $j$ to produce $y_j^0(c)$. Similarly, let $l_{j,1}(c)$ and $x_{j,1}(c)$ be the labor and intermediate inputs used by $j$’s supplier to produce $x_{j,0}(c)$. In this manner, at each step in the supply chain $c$, let $l_{j,k+1}(c)$ and $x_{j,k+1}(c)$ be the labor and intermediates used to produce $x_{j,k}(c)$. Similarly, let $\lambda_{j,k}(c)$ be the marginal social cost of the $k$th-to-last intermediate input and let $z_{j,k}(c)$ be the productivity of the technique used at the $k$th-to-last step.

If $w > 0$, an optimal allocation of labor across stages of production requires $l_{j,k+1}(c) = \alpha l_{j,k}(c)$: the labor used at each stage is a constant fraction of the labor used in the subsequent stage. To see this, note that the optimal choices of inputs for the $k$th and $k+1$th suppliers in the chain $c$ imply $\frac{\lambda_{j,k}(c)x_{j,k}(c)}{\alpha} = \frac{w l_{j,k}(c)}{1-\alpha}$ and $\frac{w l_{j,k+1}(c)}{1-\alpha} = \lambda_{j,k}(c)x_{j,k}(c)$. Total labor used in chain $c$ across all stages in production of good $j$ for final consumption, $\bar{l}_j(c)$, is then:

$$\bar{l}_j(c) \equiv \sum_{k=0}^{\infty} l_{j,k}(c) = \sum_{k=0}^{\infty} \alpha^k l_{j,0}(c) = \frac{l_{j,0}(c)}{1-\alpha}$$

The optimal choice of $l_{j,0}(c)$ implies that $\lambda_{j} y_j^0(c) = \frac{w l_{j,0}(c)}{1-\alpha}$, or $\lambda_{j} y_j^0(c) = w \bar{l}_j(c)$. Note that $y_j^0(c) > 0$ implies that $x_{j,k}(c) > 0$ and $l_{j,k}(c) > 0$ for all $k$, so that $q_{b(\phi)} = z(\phi)q_{\phi}^\alpha$ for each $\phi$ in the supply chain $c$. Thus for each $c \in \mathcal{C}_j^\infty$, $q_j \geq \prod_{k=0}^{\infty} z_{j,k}(c)\alpha^k$ with equality if $y_j^0(c) > 0$.

Let $\bar{l}_j \equiv \sum_{c \in \mathcal{C}_j^\infty} \bar{l}_j(c)$ be the total quantity of labor used to produce good $j$ for consumption. Summing across supply chains yields

$$y_j^0 = \sum_{c \in \mathcal{C}_j^\infty} y_j^0(c) = \frac{w}{\lambda_j} \sum_{c \in \mathcal{C}_j^\infty} \bar{l}_j(c) = q_j \bar{l}_j$$

(4)

With equation (4), we can derive an expression for aggregate consumption. Aggregating equation (1) for each good gives $\int_0^1 \lambda_j^{-\epsilon} dj = 1$, so that $y_j^0/Y^0 = (q_j/Q)^\epsilon$. With this, the

---

10 This is distinct from the total quantity of $j$ produced using supply chain $c$, as that might include production of intermediates for use by other firms. Because each technique has constant returns to scale, this is well defined.
labor resource constraint can then be used with equation (4) to solve for aggregate output:

\[ L = \int_0^1 \bar{l}_j \, dj = \int_0^1 \frac{y^0_j}{q^0_j} \, dj = \int_0^1 Y^0 Q^{-\varepsilon} q_j^{-1} \, dj = \frac{Y^0}{Q} \]

Lemma 1 implies that the key to solving for aggregate output is to characterize entrepreneurs’ efficiencies. Given the network of techniques, \( \Phi \), this can be done using equation (3). One approach is to look for a vector of efficiencies \( \{q_j\}_{j \in [0,1]} \) that is a fixed point of equation (3).\(^\text{11}\) However, with a continuum of firms, this is neither computationally feasible nor would it be particularly illuminating. Section 1.2 describes an approach which uses equation (3) along with the probabilistic structure of the model to sharpen the characterization of the optimal allocation.

1.2 Using the Probabilistic Structure

The set of techniques available to the different entrepreneurs, \( \Phi \), is random: for each of an entrepreneurs techniques, both its productivity and the identity of the associated supplier are randomly drawn. This section uses the law of large numbers to solve for the distribution of efficiencies that is likely to arise given this probabilistic structure. While any individual entrepreneur’s efficiency varies across realizations of the economy, the cross-sectional distribution of efficiencies does not. In particular, this section shows that the CDF of this cross-sectional distribution is the unique solution to a fixed point problem.

Let \( F(q) \) be the fraction of firms with efficiency no greater than \( q \) given \( \Phi \) and the decisions of the planner. This function is endogenous and will need to be solved for. Our strategy exploits the fact that \( F \) describes the distribution of an entrepreneur’s efficiency and that of each of its potential suppliers.

What is the probability that an entrepreneur has efficiency no greater than \( q \)? This depends on how many techniques she discovers and the efficiency each of those techniques

\(^{11}\)Taking logs of both sides gives an operator \( T \), where the \( j \)th element of \( T (\{\log q_j\}_{j \in [0,1]}) \) is \( \max_{\phi \in U_j} \{ \log z(\phi) + \alpha \log q_{s(\phi)} \} \). \( T \) satisfies monotonicity and discounting. If the support of \( z \) were bounded above, the operator would be a contraction on the appropriately bounded function space. This approach to solving the model is useful for simulations with a finite set of firms.
delivers. The number of upstream technique each firm has, $|U_j|$, follows a Poisson distribution fully described by its mean, $M$.

Two elements determine the cost-effectiveness of each technique: (i) its productivity, $z(\phi)$, drawn from an exogenous distribution $H$, and (ii) the efficiency of the supplier, $q_s(\phi)$. Since the identity of the supplier is drawn uniformly, the probability that the supplier’s efficiency is no greater than $q_s$ is $F(q_s)$.

Let $G(q)$ be the probability that the efficiency delivered by a single random technique is no greater than $q$. Recalling that the efficiency delivered by the technique is $z(\phi)q_s(\phi)$, this is:

$$G(q) = \int_{z_0}^{\infty} F\left(\frac{(q/z)^{1/\alpha}}{\alpha}\right) dH(z) \quad (5)$$

To interpret this, note that for each $z$, $F\left(\frac{(q/z)^{1/\alpha}}{\alpha}\right)$ is the portion of potential suppliers that, in combination with that $z$, leaves the firm with efficiency no greater than $q$.

Now, the probability that, given all of its techniques, a firm has efficiency no greater than $q$ is:

$$\Pr (q_j \leq q) = \sum_{n=0}^{\infty} \Pr (n \text{ techniques}) \Pr (\text{All } n \text{ techniques are } \leq q) = \sum_{n=0}^{\infty} \frac{M^n e^{-M}}{n!} G(q)^n$$

$$= e^{-M[1-G(q)]}$$

To interpret this last expression, note that if $M \left[ 1 - G(q) \right]$ is the mean of a Poisson distribution (the arrival of techniques that deliver efficiency better than $q$), then $e^{-M[1-G(q)]}$ is the probability of no such techniques.

The law of large numbers implies that with, probability one, $\Pr (q_j \leq q) = F(q)$.\footnote{The proof of Proposition 1 uses such a law of large numbers for a continuum of random variables described by Uhlig (1996). To use this, one must verify that firms’ efficiencies are pairwise uncorrelated. That is not immediately obvious in this context, as it is possible that two firms’ supply chains overlap or that one is in the other’s supply chain. However, by assumption the network is sufficiently sparse that with high probability the supply chains will not overlap: there is a continuum of firms but only a countable set of those are in any of a given firm’s potential supply chains. Therefore, for any two firms, the probability that their supply chains overlap is zero.} Using the expression for $G(q)$ from equation (5) gives a fixed point problem for the CDF of efficiency $F$:

$$F(q) = e^{-M \int_{z_0}^{\infty} [1-F\left(\frac{(q/z)^{1/\alpha}}{\alpha}\right)] dH(z)} \quad (6)$$
This recursive equation is the key to characterizing the planner’s solution.

Consider the space $\bar{F}$ of non-decreasing functions $f : \mathbb{R}^+ \mapsto [0,1]$, and consider the operator $T$ on this space defined as

$$Tf(q) \equiv e^{-M \int_0^\infty [1-f((q/z)^{1/\alpha})]dH(z)}$$

Appendix A constructively defines a subset $\mathcal{F} \subset \bar{F}$ that depends on $M$ and $H$. Proposition 1 shows that the optimal allocation is the unique fixed point of $T$ on $\mathcal{F}$. The proof, contained in Appendix A, uses Tarski’s fixed point theorem to show existence, which also provides a numerical algorithm to solve for it.

**Proposition 1.** There exists a unique fixed point of $T$ on $\mathcal{F}$, $F^{sp}$. With probability one, $F^{sp}$ is the CDF of the cross-sectional distribution of efficiencies in the solution to the planner’s problem and aggregate productivity is $Q = \left(\int_0^\infty q^{\epsilon-1}dF^{sp}(q)\right)^{1/\epsilon}$. 

The qualitative features of the economy depend on whether the average number of techniques, $M$, is greater or less than one. If $M \leq 1$, there are so few techniques that the probability that any individual firm has access to a viable supply chain is zero. This paper focuses on the more interesting case in which the average number of techniques exceeds one.

When $M$ is larger than the critical value of 1, there are at least three fixed points of the operator $T$ on $\mathcal{F}$, only one of which corresponds to the planners solution. Because of the

---

13 As shown in Appendix D, the probability that an entrepreneur has no viable chains is the smallest root $\rho$ of $\rho = e^{-M(1-\rho)}$. For $M \leq 1$, the smallest root is one, while for $M > 1$, it is strictly less than one. This uses a standard result from the theory of branching processes. If $M < 1$, $T$ is a contraction, with the unique solution $f = 1$. A phase transition when the average number of links per node crosses 1 is a typical property of random graphs, a result associated with the Erdos-Renyi Theorem. Kelly (1997) and Kelly (2005) give such a phase transition an economic interpretation.

14 There are two solutions in which $f(q)$ is constant for all $q$ which stem from the fact that equation (6) is formulated recursively. The first is $f(q) = 1$ for all $q$, which corresponds to zero efficiency (infinite marginal social cost) for all goods; if the marginal social cost of every input is infinite, then the marginal social cost of each output is infinite as well. The corresponding allocation is feasible but dominated by another feasible allocation and is therefore not the solution to the planner’s problem. A second, $f(q) = \rho \in (0,1)$, implies infinite efficiency for firms with viable supply chains and zero efficiency for those without; if inputs have zero marginal social cost, output has zero social cost. This does not correspond to a feasible allocation and is therefore not a solution to the planner’s problem.

Each of these correspond to a fixed point of equation (3). Multiple solutions to first order necessary conditions of the planner’s problem is actually a typical feature of models in which a portion of output is used simultaneously as an input, such as a standard growth model with roundabout production. If the same good enters a technological constraint as both an input and an output, the Kuhn-Tucker multiplier
multiplicity of fixed points, it is important to restrict the fixed point problem to the function space $\mathcal{F}$, a space for which there is a unique fixed point which corresponds to the planner’s allocation.

**Proposition 1** is the main technical result. It shows how cross-sectional distribution of efficiencies depends on three primitives: the distribution from which technique-specific productivities are drawn, $H$; the average number of techniques per firm, $M$; and the share of intermediate goods, $\alpha$. **Section 2** describes a decentralization of the planner’s solution. **Section 3** then uses the cross-sectional distribution to characterize how these primitives impact productivity and the organization of production.

## 2 Market Structure

Because choices of suppliers hinge on input prices, how these prices are set matters for aggregate outcomes. This section studies a market structure with monopolistic competition in sales for consumption but bilateral two-part pricing for intermediate goods. It characterizes the set of contracting arrangements that are pairwise stable—arrangements for which there are no profitable unilateral or mutually beneficial pairwise deviations.\(^{15}\)

### 2.1 Pairwise Stable Equilibrium

This section defines a pairwise stable equilibrium. This has two stages. First an arrangement determines which techniques are used and pricing for each of those techniques. Second, taking the arrangement as given, each entrepreneur sets a price for sales to the final consumer and selects its input mix.

An arrangement is a subset of techniques to be used in production and, for each of those techniques, the terms at which the buyer can purchase the associated input. These terms on that good will be on both sides of a first order condition. Consequently, the first order condition will be satisfied if the multiplier takes the value of zero or infinity. One can usually sidestep this issue by finding an alternative way to describe the production technology, i.e., solving for final output as a function of primary inputs. Much of the work in the proof of **Proposition 1** is in finding and characterizing such an alternative description of production possibilities.\(^{15}\)

\(^{15}\)As an equilibrium concept, pairwise stability is used frequently in the networks literature (Jackson (2008) provides an excellent survey).
consist of a two-part price: a price per unit, $p$, and a fixed fee, $\tau$.

**Definition 1.** An arrangement consists of: (i) A subset of techniques, $\hat{\Phi} \subseteq \Phi$, such that if $\phi', \phi \in \hat{\Phi}$ then $b(\phi) \neq b(\phi')$ (ii) For each $\phi \in \hat{\Phi}$, a two-part price, $\{p(\phi), \tau(\phi)\}$.

The first condition implies that $\hat{\Phi}$ contains at most one technique for each firm to use. The arrangement determines several objects that will be useful in characterizing an equilibrium. First, let $\hat{J}$ be the set of firms with upstream techniques in the arrangement, $\hat{J} = \{b(\phi)\}_{\phi \in \hat{\Phi}}$. Second, for each firm $j \in \hat{J}$, let $\phi_j$ denote the technique $j$ uses. Lastly, let $\hat{D}_j$ be the subset of $j$’s downstream techniques that are used, $\hat{D}_j \equiv \{\phi \in \hat{\Phi} | s(\phi) = j\}$.

In the second stage, firms make production decisions taking as given the arrangement, the wage, and the choices of others. Each $j \in \hat{J}$ chooses a price for final consumption, $p^0_j$; a quantity of labor, $l_j$; and a quantity of good $s(\phi_j)$, $x(\phi_j)$.

**Deviations:**

An arrangement is pairwise stable if there are no profitable unilateral or pairwise deviations (payoffs will be defined below). These deviations are defined as follows:

**Definition 2.** For the arrangement $\{\hat{\Phi}, \{p(\phi), \tau(\phi)\}_{\phi \in \hat{\Phi}}\}$:

- A unilateral deviation for firm $j \in [0, 1]$ is an alternative arrangement $\{\tilde{\Phi}, \{p(\phi), \tau(\phi)\}_{\phi \in \tilde{\Phi}}\}$ where $\tilde{\Phi} = \hat{\Phi} \setminus \hat{\Phi}_j$ and $\hat{\Phi}_j \subseteq (U_j \cup D_j) \cap \hat{\Phi}$ is a subset $j$’s upstream and downstream techniques.

- A pairwise deviation for the technique $\phi_0 \in \Phi$ is an alternative arrangement $\{\tilde{\Phi}, \{\tilde{p}(\phi), \tilde{\tau}(\phi)\}_{\phi \in \tilde{\Phi}}\}$ with $\tilde{\Phi} = \{\phi_0\} \cup \{\phi \in \hat{\Phi} | b(\phi) \neq b(\phi_0)\}$ and $\{p(\phi), \tau(\phi)\} = \{\tilde{p}(\phi), \tilde{\tau}(\phi)\}$ for each $\phi \in \left(\tilde{\Phi} \setminus \{\phi_0\}\right)$.

For firm $j \in [0, 1]$, a unilateral deviation is the removal from the arrangement of some subset of $j$’s upstream and downstream techniques. For example, a firm may choose not to supply goods to any of her customers and/or not to purchase inputs from her designated supplier.

A pairwise deviation for a technique is the addition to the arrangement of a new two part tariff for the technique, along with the removal from the arrangement of any other upstream technique for the buyer. There are two types of pairwise deviations. First, a firm could agree
to terms with a new supplier on a technique that is not in the arrangement. Second, a buyer and supplier could agree to new terms for a technique already in the arrangement.

**Payoffs:** Given an arrangement, define the payoff for any $j \in \hat{J}$ to be

$$\pi_j \equiv \max_{p_j^0, y_j^0, \tau_j, x(\phi_j)} \left\{ p_j^0 y_j^0 + \sum_{\phi \in D_j} [p(\phi) x(\phi) + \tau(\phi)] - [p(\phi_j) x(\phi_j) + \tau(\phi_j)] - w l_j \right\}$$

subject to the demand curve for final output and the technological constraint:

$$y_j^0 + \sum_{\phi \in D_j} x(\phi) \leq \frac{1}{\alpha^\alpha (1 - \alpha)^{1-\alpha}} z(\phi_j) x(\phi_j)^\alpha l_j^{1-\alpha}$$

Thus $j$ takes as given the arrangement, the wage, the prices for final sales set by each other firm, and the quantity demanded by each of its customers, $\{x(\phi)\}_{\phi \in D_j}$, and makes choices to maximize the sum of revenue from sales for consumption and for intermediate use net of the cost of the intermediate input and labor.

It will be useful to decompose payoffs into fixed and variable profit. Define $v_j$ to be $j$’s variable profit so that $\pi_j = v_j - \tau(\phi_j) + \sum_{\phi \in D_j} \tau(\phi)$. Similarly, given an arrangement, for any $j \in \hat{J}$ and $\phi \in (U_j \setminus \{\phi_j\})$, define $\tilde{V}_j(\phi)$ as follows: Consider the pairwise deviation for the technique $\phi$ with terms $p(\phi) = \lambda_{s(\phi)}$. Then $\tilde{V}_j(\phi)$ is $j$’s variable profit in the subsequent arrangement.

So that payoffs are well-defined out of equilibrium, we must account for an arrangement in which some $j \in \hat{J}$ that is unable to produce, because, for example, its supplier $i = s(\phi_j)$ has no way to produce ($i \not\in \hat{J}$). Such an arrangement could arise following a unilateral deviation in which $i$’s supplier removes $\phi_i$ from the arrangement.

To this end, I adopt the following convention (which will only be relevant off the equilibrium path): If the arrangement dictates that $i \in [0, 1]$ produce goods for $j \in \hat{J}$, but $i$ is unable to produce (either because $i \not\in \hat{J}$ or because $i$’s supplier $s(\phi_i)$ is unable to produce), then $i$ must compensate $j$ for lost profit, so that $j$’s payoff remains $\pi_j$. Thus while equation (7) defines the payoff and production decisions of each $j \in \hat{J}$, only the production decisions of firms for which $\hat{\Phi}$ contains a viable supply chains will be relevant for the allocations of resources.
Pairwise Stability:

Finally, we define a pairwise stable equilibrium.\(^{16}\)

**Definition 3.** A *pairwise stable equilibrium* is an arrangement \(\{\hat{\Phi}, \{p(\phi), \tau(\phi)\}_{\phi \in \hat{\Phi}}\}\), firms’ choices, \(\{p_j^0, y_j^0, x(\phi_j), l_j\}_{j \in J}\), and a wage \(w\) such that (i) Given the wage and total profit, the representative consumer maximizes utility; (ii) For each \(j \in \hat{J}\), \(\{p_j^0, y_j^0, x(\phi_j), l_j\}\) maximize \(j\)’s payoff given the arrangement, the wage, and final demand; (iii) Labor and final goods markets clear; (iv) There are no unilateral deviations that would increase a firm’s payoff or pairwise deviations that would increase one firm’s payoff without lowering the other’s.

### 2.2 Equilibrium Allocations

This section describes how pairwise stability determines the allocation of resources. Without loss of generality, we restrict attention to equilibria for which the arrangement contains a viable supply chain for each \(j \in \hat{J}\).

Given an arrangement, each firm takes factor prices as given and optimizes over inputs, so that marginal cost for firm \(j \in \hat{J}\) is \(\lambda_j \equiv \frac{1}{z(\phi_j)}p(\phi_j)\alpha w^{1-\alpha}\). If \(j \notin \hat{J}\), \(\lambda_j \equiv \infty\). Define \(q_j \equiv \frac{w}{\lambda_j}\) to be firm \(j\)’s efficiency. Proposition 2 describes implications of pairwise stability.

Appendix B shows:

**Proposition 2.** In any pairwise stable equilibrium:

1. For each technique \(\phi \in \hat{\Phi}\), price is equal to marginal cost: \(p(\phi) = \lambda_{s(\phi)}\);

2. For each technique \(\phi \in \hat{\Phi}\), the fixed fee is nonnegative: \(\tau(\phi) \geq 0\);

3. For each \(j \in \hat{J}\), \(p_j^0 = \frac{e-1}{e-1} \lambda_j\);

\(^{16}\)In closely related environments, Ostrovsky (2008) and Hatfield et al. (2012) take an alternative approach to modeling contracting arrangements. Both focus on a space of contracts that effectively dictate a price and quantity to be traded between each pair of firms, and use stronger notions of stability—that arrangements are stable with respect to deviations by larger coalitions. To see how that approach is related to the one used here, consider the following example. Suppose that according to the arrangement, \(j\) is purchasing from \(i\), and wants to deviate and purchase from \(\tilde{i}\) instead. In that deviation, for \(\tilde{i}\) to produce the goods for \(j\), she would want to use more inputs from her supplier, \(s(\tilde{\phi_j})\). With two-part pricing, the terms at which \(\tilde{i}\) purchases the additional goods from her supplier are already part of the original arrangement; she just need to place a larger order. If, instead, contracts dictated a price and quantity between each pair, this would require a new contract between \(i\) and her supplier.
4. For each $j \in [0, 1]$, $q_j = \max_{\phi \in U_j} z(\phi)q_{i(\phi)}^\alpha$ with $q_j = 0$ if $U_j$ is empty.

5. For each $j \in \hat{J}$, $\tau(\phi_j) \leq \max \left\{ v_j - \sum_{\phi \in \hat{D}_j} \tau(\phi), \max_{\phi \in (U_j \setminus \{\phi_j\})} \left\{ v_j - \tilde{V}_j(\phi) \right\} \right\}$

The first condition says that for any technique used in equilibrium, the price per unit of the input equals the supplier’s marginal cost. This follows from the two-part pricing structure. Because the productivity of a technique is specific to a particular buyer and particular supplier, there is a bilateral surplus to be split. The fixed fee provides an avenue for buyer-supplier pairs to split surplus without distorting quantities. For example, a contract in which the price per unit includes a markup and a fixed fee of zero is dominated by a contract with no markup and a fixed fee high enough to compensate the supplier for reducing her price per unit; the supplier is just as well off and the buyer will earn more from sales to the consumer. A consequence of this first condition is that production within each supply chain is efficient, as there is no double marginalization.

The second condition implies that suppliers, at worst, break even in sales of intermediate goods. The third condition says that each firm charges the usual markup in final sales given the constant elasticity of final demand.\textsuperscript{17}

The fourth condition states that the equilibrium network structure – who buys from whom – depends only on which techniques deliver the lowest marginal costs of production. This condition mirrors the necessary conditions of the planner’s problem. Finally, the fixed fee for an entrepreneur’s lowest cost technique must be low enough so that she has no incentive to deviate and either use her next best technique or drop all contracts.

One equilibrium consists of an arrangement in which marginal costs match the planner’s shadow costs and all fixed fees are zero. In that equilibrium, both the network structure of production and the allocation of goods and labor across producers match the planner’s allocation. In fact, there are many pairwise stable equilibria that decentralize the planner’s allocation. In each of these, marginal costs match the planner’s shadow costs, but fixed fees are free to vary in the range described by the second and fifth conditions. While the fixed fees determine the distribution of profit across entrepreneurs, these are simply transfers.

\textsuperscript{17}This is not immediate because it is possible for a firm to be in its own supply chain (if the supply chain is a cycle). In that case, selling more final goods would also raise sales of intermediate goods. However, since the price on sales of intermediate goods is equal to the firm’s marginal cost, selling more intermediate goods has no impact on the firm’s payoff.
Since all firms are owned by the representative consumer, total profit is independent of these transfers. As one might expect, the techniques used and the prices per unit determine the allocation of real resources. The fixed fees play no additional role.\textsuperscript{18} The markups on final sales distort the consumption–leisure margin, but since labor is supplied inelastically there is no impact on the allocation of goods and labor. If labor were elastic or if demand elasticities varied across goods, the feature that would remain intact is that production within each supply chain would remain efficient.

There is at least one equilibrium that does not decentralize the planner’s problem. This equilibrium is trivial in that \( \hat{\Phi} \) is empty so that no firms produce. This corresponds to one of constant fixed points of equation (6), as discussed in footnote (14).

The remainder of the paper restricts attention to the set of equilibria that decentralize the planner’s solution. Such a focus can be justified by appealing to an equilibrium refinement. Consider an alternative environment in which each firm has the ability to produce using only labor, with the production function \( y = qL \), with \( q \) common to all firms. Appendix B.1 shows that such an environment allows for a welfare theorem: every pairwise stable equilibrium decentralizes the planner’s solution. This result holds even when \( q \) is arbitrarily close to zero.\textsuperscript{19}

\section{Cross-Sectional Implications}

This section studies how selection of suppliers shapes cross-sectional features of the economy such as how input-output links are distributed across entrepreneurs, the size distribution, and aggregate productivity.

To illustrate these, it will be useful to focus on a particular parametric assumption that proves to be analytically tractable, allowing for closed-form expressions for the distribution of efficiencies and for aggregate output, and providing a transparent connection between the features of the environment and economic outcomes.

\textsuperscript{18}If firms made entry/exit decisions or chose the intensity with which to search for new techniques, the ex-post distribution of profit across firms would play a more central role.

\textsuperscript{19}Why does adding a labor-only option reduce the set of equilibria? The trivial equilibrium is circular in that if all of a firm’s potential suppliers cannot produce, the firm cannot produce either. The labor only option breaks this circularity.
Assume that each technique’s productivity is drawn from a Pareto distribution, \( H(z) = 1 - (z/z_0)^{-\zeta} \), with the restriction that \( \zeta > \varepsilon - 1 \). \( \zeta \) measures the thickness of the upper tail of productivity draws; a small \( \zeta \) corresponds to a thicker tail. Further, we will study the limit of a sequence of economies as \( z_0 \to 0 \). However, since the Pareto distribution collapses to zero as this minimum cutoff approaches zero, we make the following normalization to ensure that the limit is sensible. Define \( m \equiv M z_0^{\zeta} \) to be the normalized number of techniques per entrepreneur. Holding \( m \) constant, we study the limiting economy as \( z_0 \to 0 \).

In this limit, every solution \( F \) to equation (6) is the CDF of a Frechet distribution. To see this, note that \( H(z) = 1 - (z/z_0)^{-\zeta} \) and \( M = m z_0^{-\zeta} \) imply equation (5) becomes

\[
M \left[ 1 - G(q) \right] = m z_0^{-\zeta} \int_{z_0}^\infty \left[ 1 - F \left( \frac{(q/z)^{1/\alpha}}{z_0} \right) \right] \zeta z_0^\zeta z^{-\zeta - 1} dz
\]

where the second line uses the change of variables \( x = (q/z)^{1/\alpha} \). For any \( q \), as \( z_0 \to 0 \), this expression approaches \( q^{-\zeta} \) multiplied by a constant. Label this constant \( \theta \), so that equation (6) can be written as

\[
F(q) = e^{-\theta q^{-\zeta}}
\]

the CDF of a Frechet random variable. The mean of the distribution is increasing in \( \theta \), the location parameter, which was defined to satisfy \( \theta = m \int_0^\infty [1 - F(x)] \alpha x^{\zeta - 1} dx \). Using \( F(q) = e^{-\theta q^{-\zeta}} \) and integrating gives \( \theta = \Gamma(1 - \alpha) m \theta^\alpha \), or more simply

\[
\theta = \left[ \Gamma(1 - \alpha) m \right]^{\frac{1}{1 - \alpha}} \tag{8}
\]

where \( \Gamma(\cdot) \) is the gamma function.\(^{21}\)

\(^{20}\)One can compare economies with different values of \( z_0 \) (holding \( m \) fixed). In an economy with a lower \( z_0 \), each technique has stochastically lower productivity but entrepreneurs tend to have more techniques. The normalization is such that varying \( z_0 \) has no impact on the expected number of techniques at any level of productivity \( \hat{z} \), \( MH'(\hat{z}) \), as these two effects offset exactly. The only difference with a lower \( z_0 \) is that entrepreneurs draw additional relatively unproductive techniques. Note that in the limit the measure of entrepreneurs with no techniques shrinks to zero.

\(^{21}\)\( \theta = \Gamma(1 - \alpha) m \theta^\alpha \) has three non-negative roots: the one given in equation (8), zero, and infinity. The latter two correspond to the two constant fixed points of equation (6), as discussed in footnote (14).
In the expression for $F$, the exponent $\zeta$ is the same as that of the Pareto distribution $H$. This means that the distribution of efficiencies inherits the tail behavior of the productivity draws. As one might expect, $\theta$ is increasing in the (normalized) number of techniques per entrepreneur: an entrepreneur with more options to choose from will tend to have higher efficiency. The other determinant of $\theta$ is the share of intermediate goods, $\alpha$; its role will be discussed in detail below.

3.1 Stars and Superstars

To this point we have described how an entrepreneur’s efficiency depends on the arrival of upstream techniques and the availability of efficient supply chains. This section looks at techniques from the opposite perspective: how they shape demand for an entrepreneur’s good as an input. Together, these determine the entrepreneur’s size and contribution to aggregate productivity. Proposition 3 characterizes how customers are distributed across suppliers.

**Proposition 3.** Among entrepreneurs with efficiency $q$, the number of actual customers follows a Poisson distribution with mean

$$M \int_{z_0}^{\infty} \tilde{F}(zq^\alpha)dH(z)$$

where $\tilde{F}(x) \equiv \frac{\bar{F}(x)e^{-M}}{G(x)(1-e^{-M})}$ is the probability that a potential customer has no alternative techniques that deliver efficiency greater than $x$. With the functional form assumptions, this simplifies to $\frac{m}{\theta} q^{\alpha \zeta}$.

Consider an entrepreneur with efficiency $q$. $M$ is the expected number of downstream techniques. A single downstream technique with productivity $z$ would deliver efficiency of $zq^\alpha$ to its potential customer, and $\tilde{F}(zq^\alpha)$ is the probability that the potential customer has no better alternative techniques.

**Figure 2a** shows the average number of customers at each quantile in the efficiency distribution for different values of $\alpha$. As one would expect, each curve in **Figure 2a** is increasing, indicating high-efficiency entrepreneurs attract more customers.
Figure 2: Distribution of Customers

Figure 2a shows the mean number of actual customers for each quantile in the efficiency distribution. Figure 2b gives the mass of firms with $n$ customers on a log-log plot.

When the share of intermediate goods is larger, the high-efficiency firms capture an even larger share of customers. The shift is evident in Figure 2a: with higher $\alpha$, the expected number of customers is more steeply increasing with efficiency. Recall that the efficiency delivered by a single technique is $z(\phi)q^{\alpha}$. $\alpha$ determines the relative impact of each of these factors on the cost-effectiveness of a technique and, consequently, on the selection of supplier. When $\alpha$ is higher, the cost of the inputs matters more, which means the higher efficiency producers are more likely to be selected by each of their potential customers.

We next look at the unconditional distribution of customers among all firms. Proposition 4 describes the cross-sectional variation in the number of customers. If the distribution of customers were Poisson, its variance would be one. The proposition shows that the model implies a distribution with more variation.

**Proposition 4.** Among firms with viable supply chains, the distribution of customers has variance larger than one, provided equation (9) is not degenerate as a function of $q$.

The model thus provides a mechanism for higher cross-sectional variation in the distribution of links which compliments that of the influential preferential attachment model of Barabasi and Albert (1999). The preferential attachment model was designed because network models with uniform arrival of links exhibit link distributions with thin tails, whereas many real world networks feature link distributions with Pareto tails. In the preferential attachment model, there is an initial network and, over time, new links are
entrepreneurs differ in marginal cost, they also differ in their likelihood of being selected by potential customers. Some of those firms with many potential customers are able to offer lower prices and win over a larger fraction of those potential customers. While the distribution of potential customers follows a Poisson distribution, the distribution of actual customers exhibits more variation.

We now turn to the features of the environment that shape the distribution of customers. To find the mass of suppliers with $n$ customers, we simply integrate over suppliers of each efficiency. The resulting formula is given in Proposition 5, which also describes the tail of the distribution:

**Proposition 5.** Let $p_n$ be the mass of firms with $n$ customers. With the functional form assumptions, $p_n = \int_0^\infty \frac{1}{n!} \left( \frac{w^{-\alpha}}{\Gamma(1-\alpha)} \right)^n e^{-\left( \frac{w^{-\alpha}}{\Gamma(1-\alpha)} \right)} e^{-w} dw$. The counter cumulative distribution has a tail index of $1/\alpha$: $\sum_{k=n}^\infty p_k \sim \frac{1}{\Gamma(1-\alpha)/\alpha} n^{-1/\alpha}$.

Figure 2b shows the unconditional distribution of customers for different values of $\alpha$. With a larger intermediates share, the distribution has a thicker tail, as the high-efficiency firms are more likely to attract a disproportionate number of customers. In other words, when the intermediate goods share is large, the equilibrium network features more star suppliers – entrepreneurs with many customers.

Both the conditional means and unconditional distribution of customers depend on a single parameter, $\alpha$. Notably absent from in the distribution of number of customers is the parameter $\zeta$ which governs variation in firms’ efficiencies. In particular, if $\zeta$ were large there would be little variation in firms’ marginal costs, yet the distribution of customers across firms would have a Pareto tail with tail index $1/\alpha$, and lower cost firms would have a disproportionate share of customers.

---

23 Proposition 5 shows this for the unconditional distribution. To see this for the conditional distributions, let $p$ be a percentile in the distribution, so that $p = F(q) = e^{-\theta q^{-\zeta}}$. Then the average number of customers among plants at the $p$ percentile of the efficiency distribution is $m \theta q^{\alpha \zeta} = \frac{[\theta q^{-\zeta}]^{-\alpha}}{\Gamma(1-\alpha)} = \frac{[\log(1/p)]^{-\alpha}}{\Gamma(1-\alpha)}$.

24 Foster et al. (2008) provide evidence that dispersion in size is significantly larger than dispersion in marginal costs. Using data from US Census of Manufactures, they focus on industries with homogenous goods so that units and comparisons of marginal cost are meaningful. They report that the standard deviation of log physical output is four times larger than the standard deviation of log physical TFP. Because they use common materials price deflators, their measure of TFPQ is the appropriate analog to a firm’s efficiency.
Why doesn’t \( \zeta \) matter? It actually plays two offsetting roles. Aside from the share of intermediate goods, the distribution of customers depends on the relationship between variation in cost of inputs and customers’ likelihood of selecting lower-cost suppliers.\(^{25}\) The latter depends on variation in technique-specific productivity draws. If all techniques had the same productivity, each firm would select the technique with the least expensive input, and the distribution of customers would be heavily skewed toward the most efficient firms. In contrast, if all inputs had the same cost, each firm would choose the technique with the best fit (highest \( z \)), and the distribution of customers would be Poisson. Thus what matters is the level of variation in productivity draws relative to variation in suppliers’ marginal costs.

The parameter \( \zeta \) drives variation in both. When \( \zeta \) is larger so that there is less variation in productivity draws, there is also less variation in suppliers’ cost; those costs depend on the suppliers’ upstream techniques, which are themselves determined by productivity draws. With the functional forms, these two channels offset exactly in determining the distribution of customers.\(^{26}\)

Also absent is the density of the network, \( m \). While a higher \( m \) raises the number of potential customers for any firm, it also raises the number of alternatives available to each of those potential customers. Because of the regularly varying tail of productivity draws, these offset each other. In particular, the functional form assumptions allow for a separation of the level of the economy (which depends on \( m \)) from cross-sectional patterns.

### 3.2 Aggregate Output

Lemma 1 showed that aggregate output in the economy is \( Y^0 = QL \). With the functional forms, aggregate productivity is:

\[
Q = \left( \int_0^\infty q^{\zeta - 1} dF(q) \right)^{1/\zeta} = \theta^{1/\zeta} \Gamma \left( 1 - \frac{\epsilon - 1}{\zeta} \right)^{1/\epsilon} \]

\(^{25}\)Since Rosen (1981), the superstar literature studies how small differences in talent can lead to large differences in compensation. The key factor is the relationship between differences in talent and how much customers are willing to trade off differences in talent for a lower price. While the analogy is not perfect, similar forces are at play here.

\(^{26}\)As mentioned above, the distribution of efficiencies inherits the tail of the distribution of productivity draws.
Using the expression for $\theta$ from equation (8) gives an expression for final consumption:

$$Y^0 = \left[ \Gamma \left( 1 - \frac{\varepsilon - 1}{\zeta} \right) \Gamma(1 - \alpha)^{\frac{1}{1 - \alpha}} m^{\frac{1}{1 - \alpha}} \right]^{\frac{1}{\zeta}} L \tag{10}$$

There are several immediate implications. First, aggregate output is increasing in the density of the network, $m$. In an economy with more techniques, entrepreneurs tend to have larger sets of supply chains to choose from and, hence, are more likely to have found an efficient one. When $\zeta$ is small, the distribution of productivity draws has a thicker upper tail. The exponent $\frac{1}{\zeta}$ scales (exponentially) the productivity of any technique, and consequently each firm’s efficiency. The term $\Gamma \left( 1 - \frac{\varepsilon - 1}{\zeta} \right)^{\frac{1}{\zeta}}$ reflects the representative consumer’s ability to consume more of less expensive goods. $\varepsilon$, the elasticity of substitution across varieties, measures how much the final consumer is willing to substitute toward lower cost goods, while $\zeta$, the tail index of distribution of efficiencies, indicates how much cheaper these lower cost goods are.$^{27}$

Second, the exponent $\frac{1}{1 - \alpha}$ that appears in several places is the standard input-output multiplier that shows up in any model with roundabout production. The intermediate goods share determines the extent to which lower input prices feed back into lower cost of production.

A separate, more interesting, role is that $\alpha$ determines the composition of the firms supplying intermediate inputs, as illustrated by Figure 3. In each figure, the set of techniques, $\Phi$, is exactly the same. But recall from Section 3.1 that the intermediate goods share determines the likelihood that the lower-cost producers are selected as suppliers. In other words, when $\alpha$ is closer to one, the lowest cost producers are more likely to become star suppliers and be more relevant for aggregate production. At a more fundamental level, the supply chains used to produce each good are more likely to be routed through the higher-productivity techniques. Aggregate output is higher because these higher-efficiency techniques are used more intensively. Mathematically, this shows up in the term $\Gamma(1 - \alpha)$.\textsuperscript{28}

\textsuperscript{27}These terms are familiar from other studies with Frechet-distributed productivities. In the special case in which the share of intermediate goods goes to zero, input-output relationships play no role and the expression for aggregate output is the same as Kortum (1997).

\textsuperscript{28}Note that $\Gamma(x)$ is decreasing on $(0, 1)$, with $\lim_{x \to 0^+} \Gamma(x) = \infty$ and $\Gamma(1) = 1$.  

24
Figure 3: Equilibrium Supply Chains and $\alpha$

Figure 3 shows entrepreneurs’ choices of techniques. The set of techniques, $\Phi$, is held fixed; the only difference is the value of $\alpha$. The dark edges represent techniques that are used. $M = 15$ and $H(z) = 1 - z^{-2}$ for $z \geq 1$.

In sum, when $\alpha$ is high, each supplier is able to pass through cost savings to its customers at a higher rate and supply chains used in equilibrium are more likely to be routed through higher efficiency techniques.

3.3 The Distribution of Employment

The results in Section 3.1 can be used to characterize the distribution of employment. An entrepreneur’s employment depends on how much it produces for the final consumer and how much it produces for other entrepreneurs. The latter depends on both the number of actual customers and the size of each of those customers (which depends on how many customers they have, etc.). Because an entrepreneur’s employment depends on the size of each of its customers, the distribution can be characterized recursively.

Let $\mathcal{L}(\cdot)$ be CDF of the overall size distribution and let $\mathcal{L}(\cdot|q)$ be the CDF of the conditional size distribution among entrepreneurs with efficiency $q$, so that $\mathcal{L}(l) = \int_0^\infty \mathcal{L}(l|q)dF(q)$.

**Proposition 6** describes, for most parameter values, the model’s implications for these dis-
tributions. The proof is given in Appendix C.1.

**Proposition 6.** Under the functional form assumptions, the shape of the overall employment distribution depends only on $\alpha$ and $\frac{\varepsilon - 1}{\zeta}$. Suppose $\rho \equiv \min \left\{ \frac{1}{\alpha}, \frac{1}{(\varepsilon - 1)/\zeta} \right\}$ is not an integer. Then the overall and conditional distributions of employment vary regularly:

1. $1 - \mathcal{L}(l) \sim Kl^{-\rho}$

2. $1 - \mathcal{L}(l|q) \sim K^{m_\alpha \sigma \zeta} (\alpha l)^{-\rho}$

where $\rho \equiv \frac{1}{1-\alpha} \left( \frac{(1-\alpha)(1-\varepsilon+1+\alpha)}{\Gamma(1-\rho^{-1})} \right)^{\rho}$.

Proposition 6 describes the determinants of the shape of the employment distribution. The share of intermediate goods matters for the same reason as before: it determines the prevalence of star suppliers. $\frac{\varepsilon - 1}{\zeta}$ is a composite of two parameters, the elasticity of substitution across varieties in consumption and $\zeta$, the shape parameter of the distribution of efficiencies. In combination, these parameters determine the shape of the distribution of final consumption. When $\zeta$ is small, the efficiency distribution has a thicker tail, inducing a thicker tail in the distribution of final consumption. When $\varepsilon$ is high, consumers are more willing to substitute toward low cost goods, also thickening the tail of final consumption.

The first result of Proposition 6 describes the determinants of the upper tail of the size distribution. The slope of the upper tail is governed by either labor used to make intermediate inputs or by labor used to make final consumption. Across firms, the upper tail of labor used to make intermediate inputs for other firms has a Pareto tail with exponent $1/\alpha$, while labor used to make consumption has a Pareto tail of $\zeta/(\varepsilon - 1)$. The proposition says that one of these features will dominate and determine the upper tail of the overall size distribution.

Figure 4a shows how size distribution of the model compare with the size distribution among manufacturing establishments in the US.\textsuperscript{30} The data is an extract from the Statistics of US Businesses made available by Rossi-Hansberg and Wright (2007), and corresponds to the distribution of employment across manufacturing establishments in 1990.

\textsuperscript{30}This paper has no theory of the firm. Because firms are more likely to own several steps in a supply chain or produce multiple goods, products are likely a better empirical analogue to individual producers in the model. Manufacturing establishments provide the closest readily available substitute.
Figure 4: Cross-sectional Implications

Figure 4a plots the right CDF of the size distribution on a log-log plot to show the shape of the upper tail. The points labeled Data show the fraction of manufacturing establishments in 1990 with employment greater than the specified cutoff, from the SUSB. In each curve, $\alpha = 0.65$ and $L$ is chosen so that all series intersect when employment is 1000. Also shown is a line with slope $-1$, corresponding to Zipf’s law. Figure 4b plots, for various values of $\alpha$, the pairwise correlations of the logs of entrepreneurs’ efficiencies, labeled $q_b$; the productivity of the techniques they use, labeled $z$; and the efficiency of their suppliers, labeled $q_s$.

In Figure 4a, the model is parameterized so that the share of intermediate goods is $\alpha = 0.65$, roughly the cost share of intermediate inputs in the US manufacturing sector. When labor used to produce intermediates dominates the tail of the distribution ($\alpha > \frac{\varepsilon - 1}{\zeta}$), the model fits the upper tail of the establishment size distribution well regardless of the value of the composite parameter $\frac{\varepsilon - 1}{\zeta}$. The fit toward the bottom of the distribution varies, but since the model has no entry and exit, the model should not be expected to fit that feature well.

The second result of Proposition 6 says that among entrepreneurs with efficiency $q$, the distribution of employment has a power law tail with the same slope as the overall employment distribution. This stems from the input-output structure and asymmetries across customers. Among firms with efficiency $q$, the distribution of customers follows a Poisson distribution, which has a thin tail. However, these customers differ in size. An entrepreneur would be large if, for example, one of her customers is a star supplier. With the functional form assumptions, the distribution of customer size has a power law tail. The possibility of having a customer at the upper end of that distribution causes the conditional
size distribution among firms with efficiency $q$ to inherit this power law tail. The tail of the conditional size distribution is scaled by the expected number of customers, $\frac{m_{0q}^{\alpha \zeta}}{\theta}$, and by $\alpha$; when a customer uses $l$ units of labor, its supplier uses $\alpha l$ units of labor in producing the intermediate inputs.

Put differently, suppose $\alpha > \frac{\varepsilon - 1}{\zeta}$. The first result of the proposition implies that even when there is little variation in entrepreneurs’ efficiencies ($\zeta$ large) there can be a lot of variation in size in the cross-section. To use the language of Acemoglu et al. (2012), this is driven by first-order interconnections. The second result further implies that there is a lot of variation in size even among entrepreneurs with the same efficiency. While there is little variation in first-order interconnections across these entrepreneurs, second-order interconnections drive the variation in size.

### 3.4 Selection and Cross-Sectional Correlations

The model also has implications for the correlation structure between entrepreneurs’ efficiencies, the productivity of the techniques they use, and the efficiency of their suppliers. These are summarized by Proposition 7 and Figure 4b.

**Proposition 7.** Under the functional form assumptions, among techniques that are used,

1. $\text{Corr} \left( \log z(\phi), \log q_s(\phi) \right) = - \left[ 1 + \delta(\alpha)^{-1} \right]^{-1/2}$

2. $\text{Corr} \left( \log q_b(\phi), \log q_s(\phi) \right) = 0$

3. $\text{Corr} \left( \log z(\phi), \log q_b(\phi) \right) = \left[ (1 + \delta(\alpha)) (1 + 2\delta(\alpha)) \right]^{-1/2}$

where $\delta(\alpha) \equiv \frac{6\alpha^2}{\pi^2} \left( \frac{\Gamma''(1-\alpha)}{\Gamma(1-\alpha)} - \frac{\Gamma'(1-\alpha)^2}{\Gamma(1-\alpha)^2} \right)$ is positive and increasing for $\alpha \in (0, 1)$.

First, as shown in Figure 4b, entrepreneurs that use higher productivity techniques tend to buy inputs from lower efficiency suppliers. This stems from selection; an entrepreneur is more likely to select a low efficiency supplier if the technique has especially high productivity.

Second, with these functional forms, entrepreneurs’ efficiencies are uncorrelated with that of their suppliers, another manifestation of selection. Techniques associated with lower-efficiency suppliers tend to deliver lower efficiency to their potential buyers. However, low-productivity techniques that use low-efficiency suppliers are unlikely to be selected by their
potential buyers. Thus because of selection, among techniques used in equilibrium, buyers’ and suppliers’ efficiencies are uncorrelated.

Finally, Figure 4b also shows that, as one might expect, entrepreneurs with higher efficiency tend to use techniques with higher productivity. This correlation gets weaker as intermediate goods become more important. As the intermediate goods share approaches one, suppliers’ efficiencies dominate the decision of which technique to use.

The empirical work that speaks most directly to these implications was done by Atalay (2012), who provides evidence on variation in intermediate input prices and how this relates to various measures of productivity. Using data from the US Census of Manufactures from ten industries that have relatively homogeneous output and homogeneous inputs, Atalay finds, translating his results into the language of this model: (i) a negative correlation between the logs of techniques’ productivities and suppliers’ efficiencies (ii) a positive correlation between the logs of plants’ log input prices efficiency, and (iii) a positive correlation between the logs of techniques’ productivities and buyers’ efficiencies.\textsuperscript{31}

4 The Sources of Growth

This section extends the model to a dynamic environment. Time is continuous and each entrepreneur accumulates techniques. At time $t = 0$, no entrepreneur has any techniques. For each entrepreneur, upstream techniques arrive via a Poisson process with arrival rate

\textsuperscript{31}Note that the comparison is not perfect as efficiencies in the model correspond to marginal prices rather than to average prices one might observe in data. In particular, a possible explanation of (ii) is that the fixed fees a firm receives are positively correlated with the fixed fee it pays to its supplier. Such a correlation would likely result from a bargaining protocol in which suppliers received a large share of surplus. I thank Enghin Atalay for pointing this out.

Two additional confounding factors are measurement error and unobserved quality. These may be related in that unobserved quality differences mimic measurement error in prices; the model speaks to quality-corrected prices, as quality would be a feature common to all potential customers. Mis-measured input prices would induce a mechanical negative correlation between supplier’s efficiency and the match-specific productivity, while mis-measured output prices would induce a mechanical positive correlation between the buyers’ efficiency and their techniques’ productivities. In addition, to the extent that producers of higher quality output use higher quality inputs, there would be a positive correlation between buyers’ and suppliers’ efficiencies. One way to infer whether higher input prices reflect higher quality inputs or, simply, higher prices, is to check whether these higher prices are associated with good or bad outcomes for producers, a point made by Khandelwal (2010) and Kugler and Verhoogen (2012). Atalay (2012) finds that, for these industries, higher input prices are associated with fewer employees and increased likelihood of exit, suggesting quality is not the main driver of input price differences.

29
\( \mu(t) \). Over time, the network evolves as new techniques are discovered. The arrival of a technique is an expansion of an entrepreneur’s production possibilities. Entrepreneurs have perfect recall and may costlessly switch suppliers at any time. While the productivity of a technique, \( z \), is constant over time, the attractiveness of a technique varies with the cost of the associated inputs.

Under these assumptions, the cross-section of this dynamic model is exactly the same as in the static model. Letting \( M_t = \int_0^t \mu(\tilde{t})d\tilde{t} \) be the average number of techniques per entrepreneur, the cross-sectional distribution of efficiencies satisfies

\[
F_t(q) = e^{-M_t \int_0^\infty [1 - F_t((q/z)^{1/\alpha})]dH(z)}
\]

Aggregate output thus given by equation (10), so the change in aggregate output depends only on the evolution of the density of the network, \( m \):

\[
\frac{\dot{Y}_0}{Y_0} = \frac{1}{1-\alpha} \frac{\dot{m}}{\mu}
\]

These simple aggregate dynamics mask perpetual reorganization and reallocation of production. Over time, entrepreneurs discover new techniques while others substitute across techniques in response to changing input prices. The remainder of this section discusses the nature of these changes: which entrepreneurs switch suppliers, how cost savings from new techniques diffuse through the network, and how this depends on the existing network structure.

### 4.1 Dynamics of Efficiency

As an intermediate step, it will be useful to characterize the evolution of entrepreneurs’ efficiencies. The evolution of an entrepreneur’s efficiency depends on changes along her many potential supply chains. Rather than tracking individual entrepreneurs, we focus on the joint distribution of efficiency at two points in time. Just as the distribution of efficiency was the main tool in characterizing cross-sectional properties of the economy, this joint distribution will be the main tool in characterizing its dynamics. As before, the strategy is to set up a fixed point problem.32

For \( \hat{t} \geq t \), let \( F_{\hat{t},t}(q, \hat{q}) \) be the fraction of entrepreneurs with efficiency no greater than \( q \)

---

32 This method is easily extended to characterize the joint distribution at any finite number of points in time.
at time $t$ and no greater than $\hat{q}$ at $\hat{t}$. Appendix E shows that this joint distribution is a fixed point of:

$$F_{t,\hat{t}}(q, \hat{q}) = e^{-M_t \int_{z_0}^{\infty} \left[1 - F_{t,\hat{t}}\left((\frac{q}{z})^{\frac{1}{\alpha}} \cdot (\frac{\hat{q}}{z})^{\frac{1}{\alpha}}\right)\right]dH(z)} e^{-\left(M_t - M_\hat{t}\right) \int_{z_0}^{\infty} \left[1 - F_i\left(\frac{\hat{q}}{z}^{\frac{1}{\alpha}}\right)\right]dH(z)}$$

The right hand side is the product of the probabilities of two independent events: (i) that all of the entrepreneur’s upstream techniques discovered before $t$ deliver efficiency no greater than $q$ at $t$ and no greater than $\hat{q}$ at $\hat{t}$; and (ii) that all of the techniques discovered between $t$ and $\hat{t}$ deliver efficiency no greater than $\hat{q}$ at $\hat{t}$. To interpret the first term, $F_{t,\hat{t}}\left((\frac{q}{z})^{\frac{1}{\alpha}} \cdot (\frac{\hat{q}}{z})^{\frac{1}{\alpha}}\right)$ is the fraction of suppliers with efficiency low enough at $t$ and at $\hat{t}$ so that, in combination with efficiency $z$, the technique delivers efficiency no greater than $q$ at $t$ and no greater than $\hat{q}$ at $\hat{t}$. The number of techniques an entrepreneurs discovers before $t$ follows a Poisson distribution with mean $M_t$, so this first term is probability that all such techniques deliver efficiency no greater than $q$ at $t$ and no greater than $\hat{q}$ at $\hat{t}$. The second term describes techniques drawn between $t$ and $\hat{t}$. The probability that a single such technique delivers efficiency no greater than $\hat{q}$ at $\hat{t}$ is $\int_{z_0}^{\infty} F_i\left((\hat{q}/z)^{1/\alpha}\right) dH(z)$. The number of techniques discovered between $t$ and $\hat{t}$ follows a Poisson distribution with mean $M_{\hat{t}} - M_t$, so the second term is the probability of no such techniques.

### 4.2 Persistence of Vertical Relationships

Between $t$ and $\hat{t}$, an entrepreneur could keep using the same technique, switch to a technique it was not using, or switch to a newly discovered technique. This section studies how frequently these switches happen, which entrepreneurs are most likely to switch, and how these are shaped by features of the environment.

Among firms with efficiency $q$ at $t$, define the following three conditional distributions: Let $R_{t,\hat{t}}^{same}(\hat{q}|q)$ be the probability that the technique used in equilibrium at $t$ delivers efficiency no greater than $\hat{q}$ at $\hat{t}$; let $R_{t,\hat{t}}^{unused}(\hat{q}|q)$ be the probability that among techniques that were available but not used at $t$, none deliver efficiency greater than $\hat{q}$ at $\hat{t}$; and let $R_{t,\hat{t}}^{new}(\hat{q})$ be the probability that none of the techniques discovered between $t$ and $\hat{t}$ deliver efficiency greater than $\hat{q}$ at $\hat{t}$. These comprise all of the possibilities, so that $F_{t,\hat{t}}(q, \hat{q}) = \ldots$
\[ \int_{q}^{\hat{q}(q')} R_{t,t}^{\text{same}}(\hat{q}|q') R_{t,t}^{\text{unused}}(\hat{q}|q') R_{t,t}^{\text{new}}(\hat{q}) dF_t(q') \]. Appendix E.2 derives expressions for each of these conditional CDFs.

These conditional distributions can be used to characterize the fraction of entrepreneurs that switch techniques. Among entrepreneurs with efficiency \( q \) at \( t \), the hazard of switching to a newly discovered technique is

\[ \lim_{\hat{t} \to t} \frac{1}{t^2} \int_{q}^{\hat{q}(q)} R_{t,t}^{\text{same}}(\hat{q}|q) R_{t,t}^{\text{unused}}(\hat{q}|q) dR_{t,t}^{\text{new}}(\hat{q}) \]

To interpret this, \( dR_{t,t}^{\text{new}}(\hat{q}) \) is the likelihood that the best newly discovered technique delivers efficiency \( \hat{q} \), while \( R_{t,t}^{\text{same}}(\hat{q}|q) R_{t,t}^{\text{unused}}(\hat{q}|q) \) is the probability that none of the firm’s old techniques deliver efficiency greater than \( \hat{q} \) at \( \hat{t} \). The integral therefore gives the probability that a firm with efficiency \( q \) at \( t \) uses a newly discovered technique at \( \hat{t} \). Similarly, the hazard of switching to an available but unused supplier is

\[ \lim_{\hat{t} \to t} \frac{1}{t^2} \int_{q}^{\hat{q}(q)} R_{t,t}^{\text{same}}(\hat{q}|q) R_{t,t}^{\text{new}}(\hat{q}) dR_{t,t}^{\text{unused}}(\hat{q}|q) \]

With the functional form assumptions, each of these depends only on \( \alpha \) and the growth rate of the stock of techniques.

Figure 5a shows these hazards for various quantiles in the efficiency distribution. Both hazards are downward sloping: higher efficiency entrepreneurs are less likely to switch suppliers. For newly discovered techniques this is intuitive: the higher an entrepreneur’s initial efficiency, the lower the probability that a new technique delivers efficiency high enough to induce her to switch. For unused techniques, the logic is similar though less obvious, as high efficiency entrepreneurs typically have more unused techniques than low efficiency entrepreneurs.

Since larger firms tend to have higher efficiencies, we have the following two testable predictions. First, larger and higher efficiency firms will have longer relationships with their suppliers. Second, a firm is likely to have longer relationships with customers that have higher efficiency.

Each of these matches the findings of Carvalho and Gofman (2012), who use a proprietary database on supplier-customer relationships based on information from SEC filings,
press releases, websites, and other sources. They find that both higher productivity of the customer and of the supplier predict longer vertical relationships. They also find that among a supplier’s customers, the hazard of the relationship ending is lower when the customer is more productive.

When the share of intermediate goods in production is larger, the total hazard of switching is higher. The hazard of switching to a newly discovered technique is actually independent of $\alpha$; it only depends on the growth rate of the total number of techniques. In contrast, the hazard of switching to an available but unused technique is increasing in $\alpha$. The efficiency delivered by an existing technique improves only when the supplier lowers its price. When the intermediate goods share is small, any such changes have a small impact on the efficiency of the technique. This is most clear when $\alpha = 0$, an extreme case in which no firm ever switches to an unused technique. Putting these together, the overall hazard of switching is increasing in $\alpha$.

\[ \text{To see this among the whole population of firms: at } \hat{t}, \text{ the efficiency delivered by each newly discovered technique follows the same distribution as the efficiency delivered by each technique discovered prior to } \hat{t}. \text{ Each of those techniques is equally likely to be the firm’s best at } \hat{t}, \text{ so the fraction of firms using techniques discovered after } t \text{ is equal to the fraction of techniques discovered after } t. \]
4.3 Increases in Efficiency

An entrepreneur’s efficiency could increase if she switches to a new technique or if her supplier’s efficiency improves. This section asks how much of entrepreneurs’ increases in efficiency comes from each of these channels.

I will measure the average increase in efficiency among a set of entrepreneurs, \( J \subset [0, 1] \) as the decline in the price index in sales to the final consumer among those entrepreneurs’ goods relative to the decline in the aggregate price index:

\[
\frac{\ln \left( \int_{j \in J} q_j^{\varepsilon^{-1}} dJ \right)^{\frac{1}{\varepsilon-1}} - \ln \left( \int_{j \in J} q_j^{\varepsilon^{-1}} dJ \right)^{\frac{1}{\varepsilon-1}}}{\ln Q_t - \ln Q} \tag{11}
\]

Figure 5b shows how firms that switch suppliers fare relative to those that keep the same supplier when the stock of techniques grows by 10%.\(^{34}\) At all levels of efficiency, firms that switch to either new or unused techniques experience larger increases in efficiency than firms that keep the same supplier. The relationship is even stronger without conditioning on efficiency, as low efficiency firms experience larger increases and are more likely to switch.\(^{35}\) Consequently, we have the additional testable prediction that switching suppliers is associated with increases in size and efficiency. This also matches the findings of Carvalho and Gofman (2012), who find that turnover of suppliers is associated with increases in sales, employment, and labor productivity.

What portion of increases in efficiency come from each of the three sources? A convenient feature of equation (11) is that, for instantaneous changes, these increases can be cleanly decomposed into the three channels. Consider entrepreneurs with efficiency \( q \) at \( t \). Taking

\[\frac{1}{(\varepsilon - 1) \ln Q/Q} \ln \left( \frac{\int_q^\infty (\hat{q}/q)^{\varepsilon-1} R_{t,\hat{t}}^{\text{unused}}(\hat{q}|q) R_{t,\hat{t}}^{\text{new}}(\hat{q}) dR_{t,\hat{t}}^{\text{same}}(\hat{q}|q)}{\int_q^\infty R_{t,\hat{t}}^{\text{unused}}(\hat{q}|q) R_{t,\hat{t}}^{\text{new}}(\hat{q}) dR_{t,\hat{t}}^{\text{same}}(\hat{q}|q)} \right)\]

\(^{34}\)For example, consider entrepreneurs with efficiency \( q \) at \( t \) that continue to use the same supplier at \( \hat{t} \). Using equation (11), the average increase in efficiency is

\[^{35}\)When \( \hat{t} - t \) is small, the increase in efficiency conditional on switching suppliers is significantly larger than the increase among those that keep the same supplier. While the total instantaneous cost reductions from each of the three channels are of the same order of magnitude, the fraction of firms that switch to new suppliers is proportional to \( \hat{t} - t \). Since there are very few switchers when \( \hat{t} - t \) is small, the average increase in efficiency among the switchers is extremely large.
Figure 6: Increases in Efficiency

For each quantile in the efficiency distribution, Figure 6a shows average instantaneous increase in efficiency relative to the increase in aggregate productivity, decomposed into the three channels. Figure 6b shows average increase in surplus due to cost reductions in each of the three channels. In each, $\frac{\varepsilon - 1}{\xi} = \frac{1}{2}$ and $\alpha = 0.65$.

the limit as $\hat{t} \to t$ gives

$$
\lim_{t \to \hat{t}} \ln \int_{j|q_{j,t}=q} q_j^{\varepsilon - 1} dq - \ln \int_{j|q_{j,t}=q} q_j^{\varepsilon - 1} dq \ln \frac{Q_\hat{t}^{\varepsilon - 1}}{Q_t^{\varepsilon - 1}} = \lim_{t \to \hat{t}} \frac{1}{F_t(q)} \int_q^\infty \left[ \frac{(\hat{q}/q)^{\varepsilon - 1} - 1}{(Q_t/Q_\hat{t})^{\varepsilon - 1}} \right] \frac{\partial F_t(q)}{\partial \hat{q}} d\hat{q}
$$

Since the numerator is additive across entrepreneurs, we can write this as

$$
\lim_{t \to \hat{t}} \left\{ \frac{1}{F_t(q)} \int_q^\infty \left[ \frac{(\hat{q}/q)^{\varepsilon - 1} - 1}{(Q_t/Q_\hat{t})^{\varepsilon - 1}} \right] \frac{\partial F_t(q)}{\partial \hat{q}} d\hat{q} \right\}
$$

Figure 6a shows the average instantaneous increase in efficiency among firms at each quantile of the efficiency distribution. This is decomposed into the increases coming from each of the three channels.

Overall, low efficiency firms typically experience larger increases in efficiency. Consistent with Figure 5a these firms derive most of their gains from switching to either newly discovered techniques or to unused suppliers. In contrast, high efficiency firms are unlikely to find new techniques that are worth switching to, and if they do the gains tend to be small.

Notably low and high efficiency firms that keep the same supplier see, on average, the
same increases in efficiency. This stems from the fact that, given the functional forms, there are no systematic differences between suppliers of low and high efficiency firms.

4.4 Contribution to Growth

Section 4.3 showed the sources of cost reductions among firms at various levels of efficiency. This section studies the impact of these cost reductions and how they diffuse through the economy. An increase in $j$’s efficiency lowers the consumer’s cost of consuming good $j$ and also raises the efficiency of any downstream firm. We thus define a firm’s surplus to be its direct and indirect contributions to aggregate productivity.

Formally, we define firm $j$’s surplus to be the log deviation of aggregate productivity from the counterfactual in which $j$’s efficiency is set to zero. For each firm $i$, let $\tilde{q}_i$ be $i$’s efficiency in the counterfactual in which $j$’s efficiency were zero. Finally define $j$’s surplus so that $S_jdj = \frac{1}{\varepsilon - 1} \left( \ln \int_0^1 q_i^{\varepsilon - 1} di - \ln \int_0^1 \tilde{q}_i^{\varepsilon - 1} di \right)$. Suppose $J$ is the set of firms weakly downstream from $j$ (this includes $j$, $j$’s potential customers, their potential customers, etc.). Since $J$ has measure zero, we can write

$$\ln \int_0^1 \tilde{q}_i^{\varepsilon - 1} di = \ln \int_0^1 q_i^{\varepsilon - 1} di + \sum_{i \in J} \frac{q_i^{\varepsilon - 1} - \tilde{q}_i^{\varepsilon - 1}}{\int_0^1 q_i^{\varepsilon - 1} di}$$

Plugging this into the expression for $j$’s surplus gives $S_j = \frac{1}{\varepsilon - 1} \sum_{i \in J} \frac{q_i^{\varepsilon - 1} - \tilde{q}_i^{\varepsilon - 1}}{Q^{\varepsilon - 1}}$.

Let $S(q)$ be the average surplus created by firms with efficiency $q$. This can be characterized recursively:

$$S(q) = \frac{(q/Q)^{\varepsilon - 1}}{\varepsilon - 1} + M \int_{z_0}^{\infty} \int_0^{\infty} [S(\max\{\tilde{q}, zq^\alpha\}) - S(\tilde{q})] d\tilde{F}(\tilde{q})dH(z)$$

where $\tilde{F}$ is the CDF of a potential customer’s best alternative technique, derived in Proposition 3. The first term is the direct contribution to consumption. To interpret the second term, consider a single downstream technique that uses the firm’s good as an input. Suppose that a technique had productivity $z$ and was discovered by a buyer whose best other technique delivered efficiency $\tilde{q}$. With that technique, the buyer’s efficiency would be $\max\{\tilde{q}, zq^\alpha\}$, so that the expected change in the potential buyer’s surplus would be $S(\max\{\tilde{q}, zq^\alpha\}) - S(\tilde{q})$. 

36
The expected surplus from a single downstream technique can be found by integrating over the productivity of the downstream technique and the efficiency of the buyer’s best alternative technique. Lastly, the expected surplus from all downstream techniques is the expected number of downstream techniques, \( M \), multiplied by the expected surplus from each.

Appendix E.3 shows that with the functional forms there is a closed-form expression for the average surplus among firms with efficiency \( q \):

\[
S(q) = \left( \frac{q}{Q} \right)^{\varepsilon - 1} + \frac{mq^\zeta \alpha}{\theta} \frac{1}{1 - \alpha} \frac{1}{\zeta}
\]

The first term is the direct contribution to consumption, while the second term is the expected number of actual customers (given the firm’s efficiency) multiplied by the contribution to the customers’ surpluses. As in the expression for aggregate output in equation (10), the term \( \frac{1}{\zeta} \) measures the average increase in a customer’s direct contribution to consumption relative to that customer’s next best option. The multiplier \( \frac{1}{1 - \alpha} \) accounts for the fact that increasing the customer’s efficiency raises its customers’ efficiencies, etc.

Figure 6b shows the change in surplus arising from instantaneous increases in efficiency and decomposes the change into contributions from each of the three channels.\(^{36}\)

Together with Figure 6a the graph illustrates the role of the star suppliers. The highest efficiency firms have the smallest increases in efficiency, and these come almost exclusively from their suppliers lowering prices. However, these cost reductions generate the most surplus as these firms pass on their cost savings to so many customers. Thus while these star suppliers are not a source of cost savings, they play an especially large role in the diffusion of cost savings through the economy.

5 Conclusion

This paper developed a theory of the formation and evolution of an economy’s input-output architecture and characterized the implications for the organization of production and pro-

\(^{36}\)Note that the cumulative increase in surplus shown in the graph integrates to a number larger than one. This is because there is double counting. For example, a new technique might increase the efficiency of several downstream firms. The change in surplus from each increase is counted separately.
ductivity. In the model, an economy’s capacity to produce rests on relationships among producers, and aggregate productivity depends on which producers are selected to produce intermediate inputs. When intermediate goods are more important in production, the ability to charge a low price becomes more important in winning customers, and the lower-cost producers are more systematically selected as suppliers. This raises aggregate productivity and also increases the market concentration in sales of intermediate goods.

Despite the network structure, the model is analytically tractable, allowing for sharp characterizations of patterns in the cross-section and over time. While research documenting patterns of micro linkages is at an early stage, the model provides both testable implications and an organizing framework to guide empirical work as new datasets emerge.

A key channel in the model is that changes in the network structure – who buys inputs from who – matters for aggregate productivity. This channel may be useful in assessing the consequences of the marked changes in patterns of substitution across suppliers during macroeconomic crises documented by Gopinath and Neiman (2013) and Lu et al. (2013), or the impact of producers substituting to alternative suppliers in the wake of Fukushima Daiichi nuclear disaster.

The model also points to new source of misallocation. While Jones (2011, 2013) argues that distortions may cause producers to use the wrong quantities of particular inputs, here misallocation may come from an alternative channel: that distortions cause producers to use the wrong suppliers, leading them to use lower-productivity techniques or higher-cost inputs. Such distortions may include contracting frictions that cause individuals to favor forming links with family members; state mandates to purchase inputs from particular suppliers; or physical/legal barriers such as the Berlin Wall. Appropriately modified, the model would provide a natural link between such distortions and aggregate productivity.
References


Appendix

A The Solution to the Planner’s Problem

The strategy begins with defining a sequence of random variables \( \{X_N\}_{N \in \mathbb{N}} \) with the property that the maximum feasible efficiency of a firm is given by the limit of this sequence, if such a limit exists. We then show that \( X_N \) converges to a random variable \( X^{sp} \) in \( L^{\varepsilon-1} \). Next we show that the CDF of \( X^{sp} \) is the unique fixed point of \( T \) in \( \mathcal{F} \), a subset of \( \bar{\mathcal{F}} \) (and that such a fixed point exists). Letting \( F^{sp} \) be this fixed point, the law of large numbers implies the CDF of the cross-sectional distribution of efficiencies is \( F^{sp} \) and that aggregate productivity is \( \|X^{sp}\|_{\varepsilon-1} \).

A.1 Existence of a Fixed Point of Equation (6)

We begin by defining three functions, \( \bar{f} \), \( f^1 \), and \( f \), in \( \bar{\mathcal{F}} \). To do so, we define several objects that will parameterize these functions. Let \( \rho \in (0,1) \) be the smallest root of \( \rho = e^{-M(1-\rho)} \). In the definition of \( \bar{f} \), let \( \beta > \varepsilon - 1 \) be such that \( \lim_{z \to \infty} z^\beta [1 - H(z)] = 0 \). Then there exists a \( z_2 > 1 \) such that \( z > z_2 \) implies \( z^\beta [1 - H(z)] < (1 - \alpha) \). With this, define \( q_2 \) so that \( q_2^{(1-\alpha)\beta} > M \left( \frac{z_2^\beta + 1}{z_2^\beta + 1} \right) \), which also implies \( q_2^{1-\alpha} > z_2 \). In the definition of \( f^1 \), \( q_0 = \frac{1}{z_0^{1-\alpha}} \).

\[
\bar{f}(q) \equiv \begin{cases} 
\rho, & q < q_2 \\
1 - (1 - \rho) \left( \frac{q}{q_2} \right)^{-\beta}, & q \geq q_2
\end{cases}
\]

\[
f^1(q) \equiv \begin{cases} 
\rho, & q < 1 \\
1, & q \geq 1
\end{cases}
\]

\[
f(q) \equiv \begin{cases} 
\rho, & q < q_0 \\
1, & q \geq q_0
\end{cases}
\]

On \( \bar{\mathcal{F}} \), the set of right continuous, weakly increasing functions \( f : \mathbb{R}^+ \to [0,1] \), consider the partial order given by the binary relation \( \preceq \): \( f_1 \preceq f_2 \iff f_1(q) \leq f_2(q) \), \( \forall q \geq 0 \). Clearly \( \bar{f} \preceq f^1 \preceq f \).

Let \( \mathcal{F} \subset \bar{\mathcal{F}} \) be the subset of set of nondecreasing functions \( f : \mathbb{R}^+ \to [0,1] \) that satisfy \( \bar{f} \preceq f \preceq f \).

Lemma 2. \( T_f \preceq f \) and \( f \preceq T \bar{f} \)
Proof. We first show $T \bar{f} \leq \bar{f}$. For $q \geq q_0$, $T \bar{f}(q) \leq 1 = \bar{f}(q)$. For $q < q_0$:

$$T \bar{f}(q) = e^{-M \int_{q_0}^{\infty} [1 - \bar{f}((q/z)^{1/\alpha})] dH(z)} = e^{-M \int_{q_0}^{\infty} [1 - \rho] dH(z)} \leq e^{-M[1 - \rho](1 - H(q_0^{1 - \alpha}))} = \rho = \bar{f}(q)$$

We proceed to $\bar{f}$. First, for $q < q_2$, we have $T \bar{f}(q) = e^{-M \int_{0}^{\infty} [1 - \bar{f}] dH(z)} \geq e^{-M(1 - \rho)} = \rho = f(q)$.

Next, as an intermediate step, we will show that for $q \geq q_2$:

$$\int_{z_2}^{q/q_2^2} z^{\beta/\alpha} dH(z) + (q/q_2^2)^{\beta/\alpha} [1 - H(q/q_2^2)] < \left( z_2^{\beta/\alpha} + 1 \right) (q/q_2^2)^{1-\alpha} = (12)$$

To see this note we can integrate by parts to get

$$\int_{z_2}^{q/q_2^2} z^{\beta/\alpha} dH(z) + (q/q_2^2)^{\beta/\alpha} [1 - H(q/q_2^2)] + \int_{z_2}^{q/q_2^2} \frac{\beta}{\alpha} z^{\beta/\alpha - 1} [1 - H(z)] dz$$

Rearranging this gives

$$H(z_2) z_2^{\beta/\alpha} + \int_{z_2}^{q/q_2^2} z^{\beta/\alpha} dH(z) + (q/q_2^2)^{\beta/\alpha} [1 - H(q/q_2^2)] = z_2^{\beta/\alpha} + \frac{\beta}{\alpha} \int_{z_2}^{q/q_2^2} z^{\beta/\alpha - 1} [1 - H(z)] dz$$

Since $q/q_2^2 > z_2$, equation (12) follows from this and three inequalities: (i) $\int_{z_2}^{q/q_2^2} z^{\beta/\alpha} dH(z) \leq H(z_2) z_2^{\beta/\alpha}$; (ii) $z_2^{\beta/\alpha} \leq z_2^{\beta} (q/q_2^2)^{\beta/\alpha - \beta}$; and (iii) $\int_{z_2}^{q/q_2^2} z^{\beta/\alpha - 1} [1 - H(z)] dz \leq \int_{0}^{q/q_2^2} z^{\beta/\alpha - 1} [(1 - \alpha)z^{-\beta}] dz$.

Next, beginning with $1 - T \bar{f}(q) \leq -\ln T \bar{f}(q)$, we have

$$\frac{1 - T \bar{f}(q)}{1 - \rho} \leq M \int_{q_0}^{\infty} \frac{1 - \bar{f}((q/z)^{1/\alpha})}{1 - \rho} dH(z)$$

$$= M \int_{q_0}^{q/q_2^2} \frac{(q/z)^{1/\alpha}}{q_2} - q_2^{1-\alpha} dH(z) + M [1 - H(q/q_2^2)]$$

$$= \left( \frac{q}{q_2} \right)^{-\beta} M \left( z_2^{\beta/\alpha} + 1 \right) \left\{ \int_{z_2}^{q/q_2^2} \frac{z^{\beta/\alpha} dH(z) + (q/q_2^2)^{\beta/\alpha} [1 - H(q/q_2^2)]}{z^{\beta/\alpha} + 1} (q/q_2^2)^{1-\alpha} \right\}$$

$$\leq \left( \frac{q}{q_2} \right)^{-\beta} \frac{1 - \bar{f}(q)}{1 - \rho}$$

This then gives, for $q \geq q_2$, $T \bar{f}(q) \geq \bar{f}(q)$. ■

Lemma 3. There exist least and greatest fixed points of the operator $T$ in $F$, given by $\lim_{N \to \infty} T^N \bar{f}$ and $\lim_{N \to \infty} T^N f$ respectively.
**Proof.** The operator $T$ is order preserving, and $\mathcal{F}$ is a complete lattice. By the Tarski fixed point theorem, the set of fixed points of $T$ in $\mathcal{F}$ is also a complete lattice, and hence has a least and a greatest fixed point given by $\lim_{N \to \infty} T^N \bar{f}$ and $\lim_{N \to \infty} T^N \underline{f}$ respectively. ★

### A.2 Existence of a Limit

We begin with some notation. Given the set of all techniques, $\Phi$, we define several objects. A chain is a sequence of techniques (finite or infinite) $\phi_0 \phi_1 \phi_2 \ldots$ with the property $b(\phi_{k+1}) = s(\phi_k)$.

Let $\mathcal{C}(j)$ be the set of chains of techniques with the additional property that $b(\phi_0) = j$. These are the distinct chains of techniques that end with $j$. For example, if firm $j$ has access to a single technique $\phi_0$ with supplier $j_1$, and $j_1$ has access to two techniques $\phi_1$ and $\phi'_1$, then $\mathcal{C}(j)$ contains the three distinct chains $\phi_0$, $\phi_0 \phi_1$, and $\phi_0 \phi'_1$.

Let $\mathcal{C}_N(j) \subseteq \mathcal{C}(j)$ be the set of chains of length $N$ that end with $j$. In the example, $\mathcal{C}_1(j) = \{\phi_0\}$ and $\mathcal{C}_2(j) = \{\phi_0 \phi_1, \phi_0 \phi'_1\}$.

Lastly, let $\mathcal{C}^\infty(j) \subseteq \mathcal{C}(j)$ be the set of infinite chains that end with $j$. These are all of the viable supply chains to produce good $j$. $\mathcal{C}_N^\infty(j) \subseteq \mathcal{C}_N(j)$ is the set of distinct chains of length $N$ that form the beginning of an infinite supply chain. For example, $\phi_0 \phi_1 \phi_2 \ldots \in \mathcal{C}^\infty(j)$, then $\phi_0 \phi_1 \in \mathcal{C}_2^\infty(j)$.

We first show that the planner’s problem is well defined. For each chain $c \in \mathcal{C}^\infty(j)$ and each $n \geq 0$, let $z_n(c)$ be the productivity of the $n$th technique in chain $c$. In other words, for a chain $c$ of techniques $\phi_0 \phi_1 \phi_2 \ldots$ with $\phi_0$ furthest downstream, $z_n(c) = z(\phi_n)$. Finally let $q(c) \equiv \prod_{n=0}^\infty z_n(c)^{\alpha^n}$.

We want to define $q_j$ to be the efficiency provided by the most cost effective supply chain available to produce $j$, or more formally that $q_j = \sup_{c \in \mathcal{C}^\infty(j)} q(c)$. To do this, we first argue that for each $c$, $q(c)$ is well defined. The concern is that for some $c$ the sequence $\left\{\prod_{n=0}^N z_n(c)^{\alpha^n}\right\}_{N=0}^\infty$ might not converge. **Lemma 4** shows that such a sequence always converges.

Define $q_N(c) \equiv \prod_{n=0}^{N-1} z_n(c)^{\alpha^n}$ for $N \geq 1$ and $q_0(c) \equiv 1$.

**Lemma 4.** Assume $z_0 > 0$. Then for each $c \in \mathcal{C}^\infty(j)$, $\lim_{N \to \infty} q_N(c)$ exists.

**Proof.** For each $n$, $\log z_n(c)$ can be decomposed into $\log z_n^+(c) - \log z_n^-(c)$, where $z_n^+(c) = \max\{z_n(c), 0\}$ and $z_n^-(c) = \max\{-z_n(c), 0\}$, so that $\log q_N(c) = \sum_{n=0}^{N-1} \alpha^n \log z_n^+(c) - \sum_{n=0}^{N-1} \alpha^n \log z_n^-(c)$. $\sum_{n=0}^{N-1} \alpha^n \log z_n^+(c)$ is a monotone sequence so it converges to a (possibly infinite) limit. $\sum_{n=0}^{N-1} \alpha^n \log z_n^-(c)$ is a monotone sequence bounded by $\frac{\log(1/\alpha)}{1-\alpha}$ so it converges to a limit in the range $[0, \frac{\log(1/\alpha)}{1-\alpha}]$. Thus $q_N(c)$ converges to a (possibly infinite) limit. ★
If $\mathcal{C}^\infty(j)$ is non-empty, we define the random variable $X_N(j) = \max_{c \in \mathcal{C}^\infty(j)} q_N(c)$. Roughly, the remainder of this subsection shows that $X_N(j)$ converges to $q_j$. Since $q_j = \sup_{c \in \mathcal{C}^\infty(j)} \lim_{N \to \infty} q_N(c)$, we are essentially proving that the limit can be passed through the sup. A useful property of $X_N(j)$ is that its CDF is $T^N f^1$ (the logic is similar to the derivation of equation (6)).

It will also be useful to construct random variables $\bar{Y}_N(j)$ and $\overline{Y}_N(j)$ whose CDFs are $T^N f$ and $T^N \overline{f}$ respectively. Toward this, given a realization of $\Phi$, let $\{\bar{q}(c)\}_{\forall c \in \bigcup_{i=0}^\infty \mathcal{C}^\infty(j)\forall j}$ be IID random variables, each with CDF $\frac{f^{1-\rho}}{1-\rho}$. With this we define $\bar{q}_N(c) \equiv q_N(c)\bar{q}(c)^{\alpha_N}$ and $q_N(c) \equiv q_N(c)\bar{q}_0^{-\alpha_N}$ ($\bar{q}_0$ is the same as a random variable with CDF $\frac{f^{1-\rho}}{1-\rho})$. Lastly, for $N \geq 1$, let $\bar{Y}_N(j) \equiv \max_{c \in \mathcal{C}^\infty(j)} q_N(c)$ and $Y_N(j) \equiv \max_{c \in \mathcal{C}^\infty(j)} q_N(c)$. Also let $X_0(j) = 1$, $Y_0(j) = q_0$, and $\bar{Y}_0(j)$ to have CDF $\frac{f^{1-\rho}}{1-\rho}$.

If $\mathcal{C}^\infty(j)$ is empty, then $X_N(j) \equiv 0$, $\bar{Y}_N(j) \equiv 0$, $Y_N(j) \equiv 0$ for all $N \geq 0$.

To improve readability, the argument $j$ will be suppressed when not necessary.

**Lemma 5.** $\{X_N\}_{N \in \mathbb{N}}$, $\{\bar{Y}_N\}_{N \in \mathbb{N}}$, and $\{Y_N\}_{N \in \mathbb{N}}$ are uniformly integrable in $L^{\frac{\varepsilon}{1-\rho}}$.

**Proof.** First, recall that $\bar{Y}_0$ is defined so that its CDF is $\bar{f}$. Since the $T$ is order preserving, the relations $T^N f^1 \geq T^N \bar{f}$ and $T^N \bar{f} \geq T^N f^{-1} \bar{f}$ imply that $T^N f^1 \geq \bar{f}$. As a consequence, $\bar{Y}_0$ first-order stochastically dominates each $X_N$ and $\bar{Y}_N$, and, by the identical argument, $Y_N$. Therefore $\mathbb{E} |\bar{Y}_0|^{\varepsilon-1} = \frac{\bar{q}_0^{-1}}{1-\rho} < \infty$ serves as a uniform bound on each $\mathbb{E} |X_N|^{\varepsilon-1}$, $\mathbb{E} |\bar{Y}_N|^{\varepsilon-1}$, and $\mathbb{E} |Y_N|^{\varepsilon-1}$. 

**Lemma 6.** There exists a random variable $X^{sp}$ such that $X_N$ converges to $X^{sp}$ almost surely and in $L^{\frac{\varepsilon}{1-\rho}}$.

**Proof.** Let $P_N \equiv \frac{X_N}{\prod_{n=0}^\infty \mu_n}$ where $\mu_n \equiv M \int_{0}^{\infty} w^{\alpha_n} \rho^{1-H(w)} dH(w)$. We first show that $\{P_N\}$ is a submartingale with respect to $\{C_N^\infty\}$.

If $\mathcal{C}^\infty$ is empty then $\mathbb{E} [P_N | C_{N-1}^\infty] = P_{N-1} \equiv 0$. Otherwise, define a set $D_N$ as follows: Let $c^*_N \in \arg \max_{c \in \mathcal{C}^\infty} q_N(c)$ so that $X_N = q_N(c^*_N)$. Let $D_N \subseteq \mathcal{C}^\infty_N$ be the set of chains in $\mathcal{C}^\infty_N$ for which the first $N-1$ links are $c^*_{N-1}$. In other words, all chains in $D_N$ are of the form $c^*_{N-1} \phi$ for some $\phi$.

Define the random variable $D_N = \max_{c \in D_N} q_N(c)$. Since $D_N \subseteq \mathcal{C}^\infty_N$, it must be that $X_N \geq D_N$.

We now show that $\mathbb{E} [D_N | C_{N-1}^\infty] \geq \mu_N X_{N-1}$:

The probability that $|D_N| = k$ is $\frac{e^{-M(1-\rho)[M(1-\rho)]^k}}{1-e^{-M[1-\rho]^k}}$ for $k \geq 1$. To see this, note that for any node, the number of techniques is poison with mean $M$. Each of those has probability $1-\rho$ of being

\[\frac{1}{1-\rho}\] 

\[\frac{e^{-M(1-\rho)[M(1-\rho)]^k}}{1-e^{-M[1-\rho]^k}}\] 

\[\frac{e^{-M(1-\rho)[M(1-\rho)]^k}}{1-\rho}\]
viable (having a chain that continues infinitely), and we are conditioning on at least one viable
technique.

Each of those techniques has a productivity drawn from $H$. For any $\phi$ such that $c_n^* \phi \in \mathcal{D}$, we have that

$$\Pr(q_N(c_n^* \phi) < x|c_n^\infty) = \Pr(z(\phi)^{\alpha_n} < x/X_n) = H((x/X_n)^{\alpha-N})$$

Given $X_n$, if $\mathcal{D}$ consists of $k$ chains, the probability that $D_N < x$ is

$$\Pr(D_N < x|c_n^\infty, |\mathcal{D}| = k) = H((x/X_n)^{\alpha-N})^k$$

With this, the CDF of $D_N$, given $c_n^\infty$, is

$$\Pr(D_N < x|c_n^\infty) = \sum_{k=1}^{\infty} \Pr(D_N < x|X_n, |\mathcal{D}| = k) \Pr(|\mathcal{D}| = k)$$

$$= \sum_{k=1}^{\infty} H((x/X_n)^{\alpha-N})^k \frac{e^{-M[1-\rho]} [M(1-\rho)]^k}{[1-e^{-M[1-\rho]} k!}$$

$$= \frac{1}{1-e^{-M[1-\rho]}} - e^{-M[1-\rho]}$$

$$= \rho \left[1-H((x/X_n)^{\alpha-N}) \right] - \rho$$

We can now compute the conditional expectation of $D_N$ (using the change of variables $w = (x/X_n)^{\alpha-N}$):

$$\mathbb{E}[D_N|c_n^\infty] = X_n \int_{z_0}^{\infty} w^{\alpha_N} \log \rho^{-1} \frac{\mu^{1-H(w)}}{1-\rho} dH(w) = \mu_N X_n$$

Putting this together, we have

$$\mathbb{E}[P_N|c_n^\infty] = \frac{1}{\prod_{n=0}^{N} \mu_n} \mathbb{E}[X_N|c_n^\infty] \geq \frac{1}{\prod_{n=0}^{N} \mu_n} \mathbb{E}[D_N|c_n^\infty] = \frac{1}{\prod_{n=0}^{N} \mu_n} \mu_N X_n = P_{N-1}$$

We next show that $\{P_N\}$ is uniformly integrable, i.e., that $\sup_N \mathbb{E}[P_N] < \infty$. Since $\sup_N \mathbb{E}[X_N] < \infty$, it suffices to show a uniform lower bound on $\left\{ \prod_{n=0}^{N} \mu_n \right\}$. Since each $\mu_n \geq z_0^{\alpha_n}$ and $z_0 < 1$, we have that $\prod_{n=0}^{N} \mu_n \geq \prod_{n=0}^{N} z_0^{\alpha_n} \geq \prod_{n=0}^{\infty} z_0^{\alpha_n} = z_0^{-\frac{1}{\alpha}}$. 46
We have therefore established that \( \{P_N\}_{N \in \mathbb{N}} \) is a uniformly integrable (in \( L^1 \)) submartingale, so by the martingale convergence theorem, there exists an \( P \) such that \( P_N \) converges to \( P \) almost surely. By the continuous mapping theorem, there exists an \( X^{sp} \) such that \( X_N \) converges to \( X^{sp} \) almost surely. Since each \( X_N^{-1} \) is dominated by the integrable random variable \( Y_0^{-1} \), by dominated convergence we have that \( X_N \) converges to \( X^{sp} \) in \( L^1 \).

**Lemma 7.** If \( \mathcal{E}^\infty \) is nonempty then with probability one, \( X^{sp} = \sup_{c \in \mathcal{E}^\infty} q(c) \)

**Proof.** We first show that \( X^{sp} \geq \sup_{c \in \mathcal{E}^\infty} q(c) \) with probability one. Consider any realization of techniques, \( \Phi \). For any \( \nu > 0 \), there exists a \( c^* \in \mathcal{E}^\infty \) such that \( q(c^*) > \sup_{c \in \mathcal{E}^\infty} q(c) - \nu \). There also exists an \( N_1 \) such that \( N > N_1 \) implies \( q_{N}(c^*) > q(c^*) - \nu \). Lastly, with probability one there exists an \( N_2 \) such that \( N > N_2 \) implies \( X_N < X^{sp} + \nu \). We then have for \( N > \max\{N_1, N_2\} \)

\[
X^{sp} > X_N - \nu = \max_{c \in \mathcal{E}^N} q_N(c) - \nu \geq q_N(c^*) - \nu > q(c^*) - 2\nu > \sup_{c \in \mathcal{E}^\infty} q(c) - 3\nu, \quad \text{w.p.} 1
\]

This is true for any \( \nu > 0 \), so \( X^{sp} \geq \sup_{c \in \mathcal{E}^\infty} q(c) \). We next show the opposite inequality. For any \( N \), we have

\[
\sup_{c \in \mathcal{E}^\infty} q(c) \geq \sup_{c \in \mathcal{E}^\infty} q_N(c) \frac{\alpha N}{N} = X_N \frac{\alpha N}{N}
\]

Since this is true for any \( N \) and \( \lim_{N \to \infty} \left( \frac{N}{\alpha N} \right) = 1 \), we can take the limit to get \( \sup_{c \in \mathcal{E}^\infty} q(c) \geq X^{sp} \) with probability one.

**A.3 Characterization of the Limit**

We will show below that \( \log \bar{Y}_N - \log Y_N \) converges to 0 in probability. Since \( X_N \in [Y_N, \bar{Y}_N] \), it must be that both \( \bar{Y}_N \) and \( Y_N \) converge to \( X^{sp} \) in probability. Convergence in probability implies convergence in distribution, which gives two implications. First, \( T^N \bar{f} \) and \( T^N f \) converge to the same limiting function. Since these are the least and greatest fixed points of \( T \) in \( \mathcal{F} \), this limiting function, \( F^{sp} \), is the unique fixed point of \( T \) in \( \mathcal{F} \). Second, since \( T^N \bar{f} \preceq T^N f \preceq T^N f \), \( F^{sp} \) is the CDF of \( X^{sp} \).

We first show that \( \log \bar{Y}_N - \log Y_N \) converges to zero in probability.

**Lemma 8.** If \( \mathcal{E}^\infty \) is nonempty, then for any \( \eta > 1 \), \( \lim_{N \to \infty} \Pr \left( \bar{Y}_N/Y_N > \eta \right) = 0 \). 

47
Proof. Conditional on the set of supply chains $\mathcal{C}$, we have that for each chain $c$ in $\mathcal{C}_N^\infty$ that

$$q_N(c)/\bar{q}_N(c) = \left(\frac{\bar{q}(c)}{q_0}\right)^{\alpha N}.$$  We therefore have:

$$\Pr\left(\frac{q_N(c)}{\bar{q}_N(c)} \leq \eta|\mathcal{C}\right) = \Pr\left(\left(\frac{\bar{q}(c)}{q_0}\right)^{\alpha N} \leq \eta|\mathcal{C}\right) = \frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}$$

If $|\mathcal{C}_N^\infty|$ is the number of distinct chains of length $N$ that are viable, the probability that every chain $c \in \mathcal{C}_N^\infty$ satisfies $q_N(c)/\bar{q}_N(c) \leq \eta$ is $\left(\frac{f(\eta^{\alpha-N}q_0)}{1-\rho}\right)^{|\mathcal{C}_N^\infty|}$ so that

$$\Pr\left(\bar{Y}_N/Y_N \leq \eta|\mathcal{C}\right) = \left(\frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}\right)^{|\mathcal{C}_N^\infty|} \geq \left(\frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}\right)^{|\mathcal{C}_1|} \geq \left(\frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}\right)^{|\mathcal{C}_1|}$$

A standard result from the theory of branching processes (see Appendix D for a derivation) is that for any $x$, $\mathbb{E}_x^{|\mathcal{C}_1|} = \varphi^{(N)}(x)$ where $\varphi^{(N)}$ is the $N$-fold composition of $\varphi(x) = e^{-M(1-x)}$, the probability generating function for $|\mathcal{C}_1|$, and expectations are taken over realizations of $\mathcal{C}$. This implies

$$\Pr\left(\bar{Y}_N/Y_N \leq \eta\right) = \mathbb{E}_x^{|\mathcal{C}_1|} \Pr\left(\bar{Y}_N/Y_N \leq \eta|\mathcal{C}\right) \geq \mathbb{E}_x^{|\mathcal{C}_1|} \left(\frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}\right)^{|\mathcal{C}_N^\infty|} = \varphi^{(N)} \left(\frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}\right)^{|\mathcal{C}_N^\infty|}$$

Put differently, $\lim_{N \to \infty} \Pr\left(\bar{Y}_N/Y_N > \eta\right) \leq \lim_{N \to \infty} 1 - \varphi^{(N)} \left(\frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}\right)$. We complete the proof by showing $\lim_{N \to \infty} 1 - \varphi^{(N)} \left(\frac{\tilde{f}(\eta^{\alpha-N}q_0) - \rho}{1 - \rho}\right) = 0$.

To do this, we first show that for $x \in [0,1]$, $\frac{d}{dx} \varphi^{(N)}(x) \leq M^N$. To see this, note that $\varphi$ is convex and $\varphi'(1) = M$, so that $\varphi'(x) \leq M$ for $x \leq 1$. In addition, if $x \in [0,1]$ then $\varphi(x) \in (0,1]$, which implies $\varphi^{(N)}(x) \in (0,1]$ for each $N$. We then have

$$\frac{d}{dx} \varphi^{(N)}(x) = \prod_{n=1}^{N} \varphi'\left(\varphi^{(n-1)}(x)\right) \leq M^N$$
With this, for any $x$, we can bound $\varphi^{(N)}(x)$ by

$$\varphi^{(N)}(x) = \varphi^{(N)}(1) - \int_x^1 \varphi^{(N)}'(w) \, dw \geq 1 - M^N \int_x^1 dw \geq 1 - M^N [1 - x]$$

Lastly, $\lim_{N \to \infty} M^N \left[ 1 - \frac{f(q_0^N q_0^N - \rho)}{1 - \rho} \right] = \lim_{N \to \infty} M^N q_2^\beta \left( \eta^{(1 - \rho)} q_0^\beta \right) = 0$. ■

We now come to the main result.

**Proposition (1).** There is a unique fixed point of $T$ on $F$, $F^*$. $F^*$ is CDF of $X^*$. Aggregate productivity is $Q = (\int_0^\infty q^{\varepsilon - 1} dF^*(q))^{\frac{1}{\varepsilon - 1}}$ with probability one.

**Proof.** If $C^\infty$ is nonempty, the combination of log $\bar{Y}_N - \log \bar{Y}_N \xrightarrow{p} 0$, $\bar{Y}_N \geq X_N \geq Y_N$, and $X_N \xrightarrow{p} X^*$ implies that $\bar{Y}_N \xrightarrow{p} X^*$ and $Y_N \xrightarrow{p} X^*$. If $C^\infty$ is empty, then $X^* = 0$, so that $\bar{Y}_N \xrightarrow{p} X^*$ and $Y_N \xrightarrow{p} X^*$ unconditionally.

We first show that there is a unique fixed point, which is also the CDF of $X^*$. The CDFs of $\bar{Y}_N$ and $Y_N$ are $T_N^N \bar{f}$ and $T_N^N f$ respectively. The least and greatest fixed points of $T$ in $F$ are $\lim_{N \to \infty} T_N^N \bar{f}$ and $\lim_{N \to \infty} T_N^N f$ respectively. Convergence in probability implies convergence in distribution, so the least and greatest fixed point are the same, and that the fixed point is the CDF of $X^*$. Call this fixed point $F^*$.

Since $\{\bar{Y}_N\}$ and $\{Y_N\}$ are uniformly integrable in $L^{\varepsilon - 1}$, we have by Vitali’s convergence theorem that $\bar{Y}_N \to X^*$ in $L^{\varepsilon - 1}$ and $Y_N \to X^*$ in $L^{\varepsilon - 1}$.

Putting all of these pieces together, we have that the CDF of $q_j$ is $F^*$. We next show that aggregate productivity is the $Q = (\int_0^\infty q^{\varepsilon - 1} dF^*(q))^{\frac{1}{\varepsilon - 1}}$. For this we simply apply the law of large numbers for a continuum economy of Uhlig (1996). To do this, we must verify that the efficiencies are pairwise uncorrelated. This is trivial: consider two firms, $j$ and $i$. Since the set of firms in any of $j$’s supply chains is countable, the probability that $i$ and $j$ have overlapping supply chains is zero. The theorem in Uhlig (1996) also requires that the variable in question has a finite variance, and if it does, the then the $L^2$ integral exists. Here we are interested in the $L^{\varepsilon - 1}$ norm, so we require that $X^*$ is $L^{\varepsilon - 1}$ integrable. Therefore we have that $Q = (\int_0^\infty q^{\varepsilon - 1} dF^*(q))^{\frac{1}{\varepsilon - 1}}$ with probability one. ■

**B Pairwise Stable Equilibria**

We are careful here to use notation that allows for the possibility that a firm uses a production chain with a cycle: at least some of the input from $j$ is used in as an intermediate good in the supply
chain used by $j$. In such a case, if a firm decides to lower its price of final output, it sells more final output but also raises the demand for its good as an intermediate input. Given an arrangement, for each $\phi \in \hat{D}_j$, let $x(\phi, y_j^0)$ be the quantity of good $j$ sold to firm $b(\phi)$ given $y_j^0$, holding constant the final output of other firms. $x$ is non-decreasing and weakly concave in its second argument. Given an arrangement, let $\delta(p^0) \equiv Y^0 \left( p^0 / P^0 \right)^{-\varepsilon}$. The first order conditions with respect to inputs imply cost minimization so that firm $j$'s profit is:

$$\pi_j = \max_{p_j, y_j, x} \pi_j^0 y_j^0 + \sum_{\phi \in \hat{D}_j} \left[ p(\phi) x(\phi, y_j^0) + \tau(\phi) \right] - w_l - p(\phi_j) x - \tau(\phi_j)$$

subject to final demand, $y_j^0 \leq \delta(p_j^0)$, and

$$y_j^0 + \sum_{\phi \in \hat{D}_j} x(\phi, y_j^0) \leq \frac{1}{\alpha^\alpha(1-\alpha)\varepsilon} z(\phi_j) x^{\alpha} p_j^{1-\alpha}$$

The first order conditions with respect to inputs imply cost minimization so that firm $j$'s marginal cost is $\lambda_j \equiv \frac{1}{z(\phi_j)} p(\phi_j)^\alpha w^{1-\alpha}$. The price of final output $p_j^0$ then satisfies:

$$p_j^0 = \arg \max_p (p - \lambda_j) \delta(p) + \sum_{\phi \in \hat{D}_j} (p(\phi) - \lambda_j) x(\phi, \delta(p))$$

Note that if good $j$ is in the supply chain to produce good $j$ (there is a cycle), firm $j$ internalizes how changing final output would affect its sales of $j$ as an intermediate.

**Proposition (2a).** In any pairwise stable equilibrium, for any technique $\phi \in \hat{\Phi}$, then $p(\phi) = \lambda_{s(\phi)}$.

**Proof.** Assume there is a $j \in \hat{J}$ such that $p(\phi_j) \neq \lambda_{s(\phi)}$, and let $i = s(\phi)$. Consider the deviation

$$\bar{p}(\phi_j) = \lambda_i$$

$$\bar{\tau}(\phi_j) = \tau(\phi_j) + [p(\phi_j) - \lambda_i] x(\phi_j) + K$$

where $K \equiv \frac{1}{2} \left\{ \left( \bar{p}_j^0 - \bar{\lambda}_j \right) \bar{y}_j^0 - \left( \bar{p}_j^0 - \bar{\lambda}_j \right) y_j^0 + \sum_{\phi \in \hat{D}_j} \left( p(\phi) - \bar{\lambda}_j \right) \left[ x(\phi, \bar{y}_j^0) - x(\phi, y_j^0) \right] \right\}$ and $\bar{\lambda}_j = \frac{1}{z(\phi_j)} \lambda_j^\alpha w^{1-\alpha}$ is $j$'s marginal cost given the deviation. $K$ is strictly positive because $\bar{p}_j^0$ and $\bar{y}_j^0$ are optimal when $j$'s marginal cost is $\bar{\lambda}_j$. The change in $i$'s profit from the deviation is $K > 0$. The change in $j$'s profit from the deviation is

$$\bar{\pi}_j - \pi_j = \bar{p}_j^0 \bar{y}_j^0 - p_j^0 y_j^0 + \sum_{\phi \in \hat{D}_j} p(\phi) \left[ x(\phi, \bar{y}_j^0) - x(\phi, y_j^0) \right] - \left[ \bar{\lambda}_j \bar{y}_j - \lambda_j y_j \right] - \left[ \bar{\tau}(\phi_j) - \tau(\phi_j) \right]$$
Using expressions for \( \tilde{\tau}(\phi_j) \) and \( \tilde{p}(\phi_j) \) along with \( p(\phi_j)x(\phi_j) = \alpha \lambda_j y_j \) and \( \frac{\lambda_j}{\lambda_i} = \left( \frac{p(\phi_j)}{\lambda_i} \right)^\alpha \) we have

\[
\tilde{\pi}_j - \pi_j = \left( \tilde{p}_j - \tilde{\lambda}_j \right) y_j^0 - \left( p_j^0 - \tilde{\lambda}_j \right) y_j^0 + \sum_{\phi \in D_j} \left( p(\phi) - \tilde{\lambda}_j \right) \left[ x(\phi, \tilde{y}_j^0) - x(\phi, y_j^0) \right] - K
\]

\[
+ \left[ \left( 1 - \alpha \right) + \frac{\lambda_i}{p(\phi_j)} \alpha \left( \frac{p(\phi_j)}{\lambda_i} \right)^\alpha - 1 \right] \tilde{\lambda}_j y_j \]

From the definition of \( K \), the first line of the RHS is strictly positive. The second line is nonnegative because Jensen’s inequality implies \( [(1 - \alpha) + r\alpha] r^{-\alpha} \geq 1 \) for \( \alpha \in [0, 1] \) \( (r^t \) is a convex function of \( t, \) so \( (1 - \alpha)r^{-\alpha} + \alpha r^{1-\alpha} \geq r^{(1-\alpha)(-\alpha)+\alpha(1-\alpha)} = 1 \). Thus the deviation is mutually beneficial.

**Proposition (2b).** In any pairwise stable equilibrium, for any technique \( \phi \in \Phi, \tau(\phi) \geq 0. \)

**Proof.** Assume a pairwise stable equilibrium in there is a \( \phi \in \Phi \) such that \( \tau(\phi) < 0. \) Consider the unilateral deviation in which the supplier, \( s(\phi) \) removes the technique \( \phi \) from the arrangement. After the deviation, every firm’s input price remains the same except the buyer, \( b(\phi). \) The quantity produced by \( s(\phi) \) changes, as it will no longer produce inputs for \( b(\phi). \) However, since price equals marginal cost, this change in quantity produced has no effect on \( s(\phi) \)’s profit. Thus \( \tilde{\pi}_{s(\phi)} = \pi_{s(\phi)} - \tau(\phi) > \pi_{s(\phi)}, \) confirming the unilateral deviation.

**Proposition (2c).** In any pairwise stable equilibrium, for each \( j \in [0, 1], \lambda_j = \min_{\phi \in U_j} \frac{1}{z(\phi)} \lambda_{s(\phi)}^\alpha w^{1-\alpha}. \)

**Proof.** Toward a contradiction, suppose there is a \( j \) such \( \lambda_j > \min_{\phi \in U_j} \frac{1}{z(\phi)} \lambda_{s(\phi)}^\alpha w^{1-\alpha}, \) and let \( \tilde{\phi} = \arg \min_{\phi \in U_j} \frac{1}{z(\phi)} \lambda_{s(\phi)}^\alpha w^{1-\alpha}. \) Consider the deviation for \( \tilde{\phi} \) of

\[
\tilde{p}(\tilde{\phi}) = \lambda_{s(\tilde{\phi})}
\]

\[
\tilde{\tau}(\tilde{\phi}) = \frac{1}{2} \left[ \max_p \left( p - \tilde{\lambda}_j \right) \delta(p) - \max_p \left( p - \lambda_j \right) \delta(p) \right] > 0
\]

where \( \tilde{\lambda}_j \equiv \frac{1}{z(\tilde{\phi})} \lambda_{s(\tilde{\phi})}^\alpha w^{1-\alpha}. \) In the deviation, \( j \)'s marginal cost falls, but every other firm’s marginal cost is unchanged. Therefore \( j \) lowers its price of final output, but no other firm does, so the quantity of final output of \( j \) rises.

Firm \( j \) finds the deviation profitable, \( \tilde{\pi}_j > \pi_j, \) because marginal cost has fallen: (i) \( j \) now makes a positive profit on each unit of output sold as intermediate output (if \( j \) is in a cycle, then profit from intermediates would rise even more); (ii) The change in profit from final sales is greater than the fee \( \tilde{\tau}(\tilde{\phi}); \) and (iii) \( j \) no longer has to pay \( \tau(\phi_j) \) (which is nonnegative). Firm \( s(\tilde{\phi}) \) also finds the deviation profitable, \( \tilde{\pi}_{s(\tilde{\phi})} > \pi_{s(\tilde{\phi})}, \) because it now collects \( \tilde{\tau}(s(\tilde{\phi})). \) Since price equals marginal
Proposition (2d). In any pairwise stable equilibrium, for each \( j \in \hat{J} \),

\[
\tau(\phi_j) \leq \max \left\{ v_j - \sum_{\phi \in \hat{D}_j} \tau(\phi), \max_{\phi \in (U_j \setminus \{\phi_j\})} \left\{ v_j - \tilde{V}_j(\phi) \right\} \right\}
\]

Proof. We first consider pairwise deviations. Pairwise stability implies that \( p(\phi_j) = \lambda_{s(\phi_j)} \). Any alternative two part price in which \( p(\phi_j) = \lambda_{s(\phi_j)} \) would leave the sum of the buyer’s and supplier’s payoffs unaltered, and any alternative price per unit would strictly decrease the sum of their payoffs. Thus any such deviation cannot increase one’s profit without diminishing the other’s.

Now consider a pairwise deviation for the technique \( \phi \in (U_j \setminus \{\phi_j\}) \). Without the deviation, the sum of the buyer’s and supplier’s payoffs is \( \pi_{s(\phi)} + \left( v_j - \tau(\phi_j) + \sum_{\phi \in \hat{D}_j} \tau(\phi) \right) \). After the deviation the sum of their payoffs is \( \pi_{s(\phi)} + \left( \tilde{V}_j(\phi) + \sum_{\phi \in \hat{D}_j} \tau(\phi) \right) \). Thus the deviation strictly increases one’s profit without diminishing the other’s only if

\[
v_j - \tau(\phi_j) < \tilde{V}_j(\phi)
\]

Consider now a unilateral deviation. For any pairwise stable arrangement, price equals marginal cost and the fixed fee is non-negative. Thus any unilateral deviation in which \( j \) drops a subset of customers cannot strictly increase \( j \)'s payoff.

Next, each firm can guarantee itself a payoff of zero by unilaterally dropping all contracts, so it must be that in any pairwise stable equilibrium all firms have non-negative payoff. As a consequence, if \( j \) unilaterally drops its supplier, it cannot have a strictly positive payoff, as it will not produce and it must compensate any downstream firms. Thus the best unilateral deviation in which \( j \) drops its supplier is the one in which \( j \) drops all contracts, which gives it a payoff of zero. Thus \( j \) would unilaterally drop its supplier only if

\[
v_j - \tau(\phi_j) + \sum_{\phi \in \hat{D}_j} \tau(\phi) < 0
\]
B.1 Labor Only Option

We now show that if each firm has access to a production technology uses only labor, \( y_j = qL \) with common productivity \( q \), then in every pairwise stable equilibrium aggregate productivity \( Q \) is the same as in the planner’s problem.

Let \( q_{pw}^j \) be the efficiency of firm \( j \) in a pairwise stable equilibrium, and let \( F_{pw}(q) \) be the fraction of firms in that equilibrium with efficiency no greater than \( q \). We can compare this to the CDF of efficiencies that solve the planner’s problem, \( F_{sp} \).

**Proposition 8.** In any pairwise stable equilibrium, \( F_{pw} = F_{sp} \) and \( Q_{pw} = Q_{sp} \).

**Sketch of Proof.** In both the planner’s problem and in any pairwise stable equilibrium, each firm chooses the technique that delivers the most cost effective combination:

\[
q_j = \max \left\{ q, \max_{\phi \in U_j} \left\{ z(\phi)q^\alpha s(\phi) \right\} \right\}
\]

For each firm \( j \), we are interested in the efficiency in the social planner’s problem, \( q_{sp}^j \), and efficiency in a pairwise equilibrium. These satisfy the following three equations:

\[
q_{sp}^j = \max \left\{ \sup_{c \in C^\infty(j)} q(c), \sup_{N \in N} \max_{c \in C_N(j)} q_N(c)q^{\alpha N} \right\}, \quad \text{a.e. } j \in [0, 1]
\]
\[
q_{pw}^j \geq \sup_{N \in N} \max_{c \in C_N(j)} q_N(c)q^{\alpha N}, \quad \forall j \in [0, 1]
\]
\[
q_{pw}^j \leq q_{sp}^j, \quad \text{a.e. } j \in [0, 1]
\]

These can be interpreted as follows: First, the chain with the maximum feasible efficiency available to the planner is either an infinite chain or a finite chain that ends with the furthest firm upstream using the outside option. Second, in any pairwise stable equilibrium, the efficiency of a firm must be at least that of any finite chain that ends with a firm using its outside option \( q \). If otherwise, at least one of the firms in that chain must be using a suboptimal technique. Third, efficiency in a pairwise stable equilibrium must be feasible. The first and third statements are satisfied for almost every firm because the planner may deviate from the maximum feasible efficiency for a set of firms with measure zero.

The remainder of the proof follows broadly along the same lines as Appendix A with some minor modifications. As a result I will only provide a sketch, pointing out the differences. We can
construct an operator in the same manner as before:

\[
Tf(q) \equiv \begin{cases} 
0, & q < q_e \\
e^{-M \int_0^\infty f\left(\frac{x}{z}\right) dH(z)}, & q \geq q_e
\end{cases}
\]

We can also define three functions.

\[
f(q) \equiv \begin{cases} 
0, & q < q_1 \\
1, & q \geq q_1
\end{cases}
\]

\[
f^1(q) \equiv \begin{cases} 
0, & q < 1 \\
1, & q \geq 1
\end{cases}
\]

\[
\tilde{f}(q) \equiv \begin{cases} 
0, & q < q_2 \\
1 - \left(\frac{q}{q_2}\right)^{-\beta}, & q \geq q_2
\end{cases}
\]

With the additional restriction that \(q_2 > q_e\).

Let \(\mathcal{F}\) be the subset of \(\tilde{\mathcal{F}}\) defined by \(\{f|\tilde{f} \preceq f \preceq \bar{f}\}\). In the same manner as before, we can show that \(T\tilde{f} \succeq \bar{f}\) and \(Tf \preceq \bar{f}\). This means that there is a least and greatest fixed point of \(T\) in \(\mathcal{F}\), given by \(\lim_{N \to \infty} T^N \tilde{f}\) and \(\lim_{N \to \infty} T^N f\) respectively.

Define \(\gamma_N \equiv \max_{N \leq N} \max_{c \in C} \prod_{n=0}^{N-1} q(c)^{\alpha^n}\). With this we can define variables \(X_N \equiv \max\{\gamma_N, \max_{c \in C} \prod_{n=0}^{N-1} q(c)^{\alpha^n}\}\) and \(\bar{Y}_N \equiv \max\{\gamma_N, \max_{c \in C} \prod_{n=0}^{N-1} q(c)^{\bar{q}(c)^{\alpha^n}}\}\), where \(\bar{q}(c)\) is a random variable drawn independently for each \(c \in C\) for each realization of \(\Phi\), each with CDF \(\tilde{f}\).

The CDFs of \(X_N\), \(\bar{Y}_N\), and \(\gamma_N\) are \(T^N f^1\), \(T^N \bar{f}\), and \(T^N \tilde{f}\) respectively.

Let \(X^{sp} \equiv \lim_{N \to \infty} X_N\) and \(X^{pw} \equiv \lim_{N \to \infty} \bar{Y}_N\). In the same manner as before we can show that these limits exists and that with probability 1 we have both \(X^{sp} = q^{sp}\) and \(X^{pw} \leq q^{pw} \leq X^{sp}\). Lastly, we can show that \(F^{sp}\) is the unique fixed point of \(T\) on \(\mathcal{F}\), and is the CDF of both \(X^{sp}\) and \(X^{pw}\). Applying the law of large numbers gives \(Q^{sp} = Q^{pw}\). \(\blacksquare\)

**C Distribution of Customers**

**Proof of Proposition 3.** We first derive an expression for \(\tilde{F}(x)\). This is the probability that a potential buyer has no alternative techniques that deliver efficiency better than \(x\). The potential buyer will have \(n - 1\) other techniques with probability \(e^{-M \frac{M^n}{n(1-e^{-M})}}\). The probability that a single alternative delivers efficiency no greater than \(x\) is \(G(x)\). Therefore the probability that none of the
potential buyer’s alternatives deliver efficiency better than $x$ is:

$$
\hat{F}(x) = \sum_{n=1}^{\infty} \frac{e^{-xM} M^n G(x)^n}{1 - e^{-M}} = \frac{1}{G(x)(1 - e^{-M})} \left[ \sum_{n=0}^{\infty} \frac{e^{-xM} M^n}{n!} G(x)^n - e^{-xM} \right] = \frac{F(x) - e^{-M}}{G(x)(1 - e^{-M})}
$$

Consider an entrepreneur with efficiency $q_s$. If a single downstream technique has productivity $z$, the technique delivers efficiency $zq_s$ to the potential customer, and will be selected by that customer with probability $\hat{F}(zq_s^\alpha)$. Integrating over possible productivities, the probability that a single downstream technique is used by the customer is

$$
\int_{z_0}^{\infty} \hat{F}(zq_s^\alpha) dH(z).
$$

Since the number of downstream techniques is Poisson with mean $M$, the number of downstream techniques that are used is Poisson with mean $M \int_{z_0}^{\infty} \hat{F}(zq_s^\alpha) dH(z)$.

Using the functional form for $H$ and taking the limit as $z_0 \to 0$ this can be simplified considerably. Since $\lim_{z_0 \to 0} e^{-mz_0^\zeta} = 0$ and $\lim_{z_0 \to 0} G(q) = 1$, we have that

$$
\lim_{z_0 \to 0} m z_0^{-\zeta} \int_{z_0}^{\infty} \frac{F(zq_s^\alpha) - e^{-mz_0^\zeta}}{G(zq_s^\alpha) - e^{-mz_0^\zeta}} z^{-\zeta-1} dz = m \int_{0}^{\infty} e^{-\theta(q_s^\alpha)}^{-\zeta} z^{-\zeta-1} dz = \frac{m}{\theta} q_s^\alpha.
$$

**Proof of Proposition 4.** By the law of total variances, the variance of actual customers is

$$
\text{var} \ (\text{customers}) = \mathbb{E} \left[ \text{var} \ (\text{customers}|q) \right] + \text{var} \left( \mathbb{E} [\text{customers}|q] \right)
$$

We now show that the first term equals one. Among entrepreneurs with efficiency $q$, the distribution of actual customers is Poisson with mean and variance $M \int_{z_0}^{\infty} \hat{F}(zq_s^\alpha) dH(z)$. Thus the expectation of the variances is equal to the expectation of the means:

$$
\mathbb{E} [\text{var} \ (\text{customers}|q)] = \mathbb{E} \left[ \mathbb{E} (\text{customers}|q) \right] = 1
$$

Thus if the second term is positive, which is the case if equation (9) is not degenerate as a function of $q$, then the overall variance of customers is greater than 1. ■

**Proof of Proposition 5.**

With the functional forms, among firms with efficiency $q$, the distribution of customers is Poisson with mean $\frac{m}{\theta} q^\alpha$. Integrating across efficiencies, the unconditional probability that a firm has $n$
The second equality uses $\theta = \Gamma(1 - \alpha) m \theta^\alpha$. This can be written as

$$p_n = \int_0^\infty u^n e^{-u} \frac{e^{-[\Gamma(1-\alpha)u]^{-1/\alpha}}}{\Gamma(1 - \alpha)^{2/\alpha}} u^{-\frac{1}{\alpha} - 1} du$$

Since $\lim_{u \to \infty} e^{-[\Gamma(1-\alpha)u]^{-1/\alpha}} = 1$, theorem 2.1 of Willmot (1990) gives $\lim_{n \to \infty} \frac{n^{-1/\alpha} L\Sigma_{k=n}^{p_n} \Gamma(1 - \alpha)^{-1/\alpha} = 1}$ Then theorem 1 of Feller (1971) VIII.9 implies that $\lim_{n \to \infty} n p_n^{\sum_{k=n}^{p_n} p_k} = \frac{1}{\alpha}$, giving the desired result.

### C.1 The Distribution of Employment

Because employment is the sum of several components—labor used to produce inputs for each customer and for the final consumer—rather than working with the CDF of the size distribution, $L(\cdot)$, it will be easier to work with its Laplace-Stieltjes transform, $\hat{L}(s) \equiv \int_0^\infty e^{-sl} dL(l)$. Similarly, if $L(\cdot|q)$ is CDF of the conditional size distribution among firms with efficiency $q$, its transform is $\hat{L}(s|q) \equiv \int_0^\infty e^{-sl} dL(l|q)$. These are related in that $L(l) = \int_0^\infty L(l|q)dF(q)$ and $\hat{L}(s) = \int_0^\infty \hat{L}(s|q)dF(q)$. This section characterizes these transforms and then studies their implications for the size distribution.

We first derive a the relationship between the conditional size distributions among firms with different efficiencies. Recall that $\tilde{F}(q) \equiv \frac{F(q) - e^{-M}}{G(q)[1 - e^{-M}]}$ describes the CDF of a potential buyer’s best alternative technique.

**Lemma 9.** The transforms $\{\hat{L}(|q)\}$ satisfy

$$\hat{L}(s|q) = e^{-s(1-\alpha)(q/Q)^{\varepsilon-1} L} e^{-M \int_0^\infty \tilde{F}(zq^\alpha) \frac{[1 - \hat{L}(zq^\alpha)]dH(z)}{G(z)}}$$

**Proof.** Total labor used by a firm is the sum of labor used to make output for consumption and for use as an intermediate input. We use the fact that the Laplace-Stieltjes transform of a sum of random variables is the product of the transforms of each.

A firm with efficiency $q$ uses $(1 - \alpha)(q/Q)^{\varepsilon-1} L$ units of labor in making goods for final con-
assumption. The transform of this is \( e^{-s(1-\eta)(q/Q)^{\frac{1}{\eta}} - 1}_L \).

We next consider labor used to make intermediate inputs. Recall from the proof of Lemma 1 the convenient fact that if firm \( j \) uses \( l_j \) units of labor, \( j \)’s supplier will use \( \alpha l_j \) units of labor to make the inputs for \( j \). Thus if the transform of labor used by a buyer with efficiency \( q_b \) is \( \hat{L}(s|q_b) \), then the transform of labor used by its supplier to make intermediates is

\[
\int_0^\infty \frac{1}{\alpha} \Pr \left( l_j = \frac{l}{\alpha} \right) e^{-\alpha t} \, dt = \int_0^\infty \Pr \left( l_j = \frac{l}{\alpha} \right) e^{-\alpha(\alpha s) \frac{1}{\alpha} d} \left( \frac{l}{\alpha} \right) = \hat{L}(\alpha s|q_b)
\]

For a firm with efficiency \( q \), consider a single downstream technique with productivity \( z \), so that the technique delivers efficiency to the buyer of \( zq^\alpha \). With probability \( \tilde{F}(zq^\alpha) \) it is the buyer’s best technique, in which case the transform of labor used to create intermediates for that customer is \( \hat{L}(\alpha s|zq^\alpha) \). With probability \( 1 - \tilde{F}(zq^\alpha) \) the potential buyer uses an alternative supplier, in which case the transform of labor used to create intermediates for that customer is simply 1. Putting these together and integrating over possible realizations of productivity, the transform of labor used to make intermediates for a single potential customer is

\[
\int_0^\infty \left\{ \left[ 1 - \tilde{F}(zq^\alpha) \right] + \tilde{F}(zq^\alpha) \hat{L}(\alpha s|zq^\alpha) \right\} dH(z) = 1 - \int_0^\infty \tilde{F}(zq^\alpha) \left[ 1 - \hat{L}(\alpha s|zq^\alpha) \right] dH(z)
\]

Each firm has \( n \) potential customers with probability \( \frac{M^n e^{-M}}{n!} \), so the transform over labor used to create all intermediate goods (summing across all potential customers) is

\[
\sum_{n=0}^\infty \frac{M^n e^{-M}}{n!} \left( 1 - \int_0^\infty \tilde{F}(zq^\alpha) \left[ 1 - \hat{L}(\alpha s|zq^\alpha) \right] dH(z) \right)^n = e^{-M \int_0^\infty \tilde{F}(zq^\alpha) [1 - \hat{L}(\alpha s|zq^\alpha)] dH(z)}
\]

\( \hat{L}(s|q) \) is simply the product of the transforms of labor used for to make final consumption and labor used to make intermediate inputs.

Using the functional forms, we can get at the overall size distribution without the intermediate step of solving for the conditional size distributions.

**Lemma 10.** Define \( v \equiv \frac{\eta - 1}{\eta} \). With the functional form assumptions, the transform \( \hat{L}(\cdot) \) satisfies

\[
\hat{L}(s) = e^{-s(1-\eta) \frac{1}{(1-\eta)} - 1}_L e^{-\eta \frac{1}{(1-\eta)} \left[ 1 - \hat{L}(\alpha s) \right]} e^{-t} \, dt
\]

**Proof.** First, using the functional forms, the term \( M \int_0^\infty \tilde{F}(zq^\alpha) \left[ 1 - \hat{L}(\alpha s|zq^\alpha) \right] dH(z) \) can be written as \( mzq^{\alpha} \int_0^\infty \tilde{F}(zq^\alpha) \left[ 1 - \hat{L}(\alpha s|zq^\alpha) \right] \zeta z^{\alpha} e^{-z} \, dz \). Since \( \tilde{F}(zq^\alpha) \to e^{\theta(zq^\alpha)^{\beta}} \), this becomes
interested in the Tauberian theorem of Bingham and Doney (1974), this will imply that
\[ 1 - \hat{L}(\alpha s|w) \]
formula (a generalization of the chain rule to higher derivatives), we have that
\[ \rho \]
strategy is to show that \( \mu \)
\[ \alpha \]
\[ \epsilon \]
An immediate consequence of Lemma 10 is that the shape of the size distribution depends only
\[ \alpha \]
and \( \xi \)

### C.2 Tail Behavior

Let \( \rho = \min \{ \alpha^{-1}, v^{-1} \} \) and let \( N \) be the greatest integer that is strictly less than \( \rho \). For any
integer \( n \), let \( \mu_n = \int_0^\infty t^n dL(l) = (-1)^n \hat{L}^{(n)}(0) \) be the \( n \)th moment of the size distribution. The
strategy is to show that \( \mu_N - (-1)^N \hat{L}^{(N)}(s) \) is regularly varying with index \( \rho - N \) as \( s \searrow 0 \). Using
the Tauberian theorem of Bingham and Doney (1974), this will imply that \( 1 - L(l) \) is regularly
varying with index \( -\rho \) as \( l \to \infty \). The theorem gives this implication only when \( 0 < \rho - N < 1 \), so we restrict attention to that case.

Define \( \varphi(s; t) = s \frac{1}{(1-v)^{(1-v)}} t^{-v} + \frac{1}{(1-\alpha)^{(1-\alpha)}} t^{-\alpha} \) so that \( \hat{L}(s) = \int_0^\infty e^{-\varphi(s;t)} e^{-t} dt \). Since we will be
interested in \( \hat{L}^{(n)} \), it will be useful to derive an expression for \( \frac{d^n}{ds^n} [e^{-\varphi(s;t)}] \). By Faa di Bruno’s
formula (a generalization of the chain rule to higher derivatives), we have that
\[
\frac{d^n}{ds^n} [e^{-\varphi(s;t)}] = e^{-\varphi(s;t)} \sum_{\tau \in \mathcal{I}_n} \frac{n!}{\tau_1!(1)_{\tau_1} \cdots \tau_n!(n)_{\tau_n}} \prod_{j=1}^n [-\varphi^{(j)}(s; t)]^{\tau_j}
\]
where \( \mathcal{I}_n \) is the set of all \( n \)-tuples of non-negative integers \( \tau = (\tau_1, \ldots, \tau_n) \) such that \( 1\tau_1 + \cdots + n\tau_n = n \).

Since \( \varphi^{(1)}(s; t) = \frac{(1-v)^{(1-v)}}{(1-v)^{(1-v)}} t^{-v} + \frac{\alpha}{(1-\alpha)^{(1-\alpha)}} t^{-\alpha} \) and \( \varphi^{(j)}(s; t) = -\frac{\alpha}{(1-\alpha)^{(1-\alpha)}} t^{-\alpha} \) for \( j \geq 2 \), we have

\[
\frac{d^n}{ds^n} [e^{-\varphi(s;t)}] = e^{-\varphi(s;t)} \sum_{\tau \in \mathcal{I}_n} \frac{n!}{\tau_1!(1)_{\tau_1} \cdots \tau_n!(n)_{\tau_n}} \prod_{j=1}^n [-\varphi^{(j)}(s; t)]^{\tau_j}
\]
Thus we can write
\[
\prod_{j=1}^{n} \left[ -\varphi^{(j)}(s; t) \right]^{\ell_j} = \sum_{k=0}^{n-1} \binom{n}{k} \left[ -\frac{(1 - \alpha)}{(1 - v)} t^{-v} \right]^{k} \left[ \frac{\alpha \hat{L}^{(1)}(\alpha s)}{\Gamma(1 - \alpha)} t^{-\alpha} \right]^{\ell_1 - k} \prod_{j=2}^{n} \frac{\alpha^j \hat{L}^{(j)}(\alpha s)}{\Gamma(1 - \alpha)} t^{-\alpha} \]
\[
= \sum_{k=0}^{n-1} \binom{n}{k} \left[ -\frac{(1 - \alpha)}{(1 - v)} \frac{\Gamma(1 - \alpha)}{\alpha \hat{L}^{(1)}(\alpha s)} \right]^{k} \prod_{j=1}^{n} \frac{\alpha^j \hat{L}^{(j)}(\alpha s)}{\Gamma(1 - \alpha)} t^{-\alpha} \]
\[
= \sum_{k=0}^{n-1} \binom{n}{k} \left[ -\frac{(1 - \alpha)}{(1 - v)} \frac{\Gamma(1 - \alpha)}{\alpha \hat{L}^{(1)}(\alpha s)} \right]^{k} \prod_{j=1}^{n} \frac{\alpha^j \hat{L}^{(j)}(\alpha s)}{\Gamma(1 - \alpha)} t^{-\alpha} \]
\[
\text{Thus we can write}
\]
\[
\frac{d^n}{ds^n} \left[ e^{-\varphi(s; t)} \right] = e^{-\varphi(s; t)} \sum_{k=0}^{n} \sum_{\ell \in I_n} B_{n,k}(s) t^{-\beta(\ell, k)}
\]

where
\[
B_{n,k}(s) = \frac{n!}{\ell_1!(\ell_1)! \ldots !n!(\ell_n)!} \left[ -\frac{(1 - \alpha)}{(1 - v)} \frac{\Gamma(1 - \alpha)}{\alpha \hat{L}^{(1)}(\alpha s)} \right]^{k} \prod_{j=1}^{n} \frac{\alpha^j \hat{L}^{(j)}(\alpha s)}{\Gamma(1 - \alpha)} t^{-\alpha} \]
\[
\beta(\ell, k) = \alpha \left( \sum_{j=1}^{n} \ell_j - k \right) + v k
\]

Note that if $n < \rho$ then each $\beta(\ell, k) \in (0, 1)$. We first show that the $n$th derivative can be taken inside the integral

**Lemma 11.** For any integer $n < \rho$, $\hat{L}^{(n)}(s) = \int_0^\infty \frac{d^n}{ds^n} \left[ e^{-\varphi(s; t)} \right] e^{-t} dt$.

**Proof.** We first show that for each $\ell \in I_{n-1}$,

\[
\frac{d}{ds} \int_0^\infty e^{-\varphi(s; t)} t^{-\beta(\ell, k)} e^{-t} dt = \int_0^\infty \frac{d}{ds} \left[ e^{-\varphi(s; t)} \right] t^{-\beta(\ell, k)} e^{-t} dt
\]

Since $\frac{d}{ds} \left[ e^{-\varphi(s; t)} \right] = \left( \frac{(1 - \alpha)}{(1 - v)} t^{-v} + \frac{-\alpha \hat{L}^{(1)}(\alpha s)}{\Gamma(1 - \alpha)} t^{-\alpha} \right) e^{-\varphi(s; t)}$, the integrand on RHS is dominated by $\left( \frac{(1 - \alpha)}{(1 - v)} t^{-v} + \frac{-\alpha \hat{L}^{(1)}(\alpha s)}{\Gamma(1 - \alpha)} t^{-\alpha} \right) t^{-\beta(\ell, k)} e^{-t}$, which is integrable for each $s$ because $\alpha + \beta(\ell, k) < 1$ and $v + \beta(\ell, k) < 1$.

With this, we proceed by induction. Trivially, the conclusion holds for $n = 0$. Now for $n < \rho$, assume that $\hat{L}^{(n-1)}(s) = \int_0^\infty \frac{d^{n-1}}{ds^{n-1}} \left[ e^{-\varphi(s; t)} \right] e^{-t} dt$. Equation (13) implies

\[
\hat{L}^{(n-1)}(s) = \sum_{\ell \in I_{n-1}} \sum_{k=0}^{\ell_1} B_{n-1,k}(s) \int_0^\infty e^{-\varphi(s; t)} t^{-\beta(\ell, k)} e^{-t} dt
\]
Differentiating each side gives

\[
\hat{L}^{(n)}(s) = \sum_{i \in I_n} \sum_{k=0}^{l_i} dB_{n-1,i,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(i,k)} e^{-t} dt + B_{n-1,i,k}(s) \frac{d}{ds} \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(i,k)} e^{-t} dt
\]

\[
= \int_0^\infty \sum_{i \in I_n} \sum_{k=0}^{l_i} \left[ dB_{n-1,i,k}(s) e^{-\varphi(s;t)} + B_{n-1,i,k}(s) \frac{d}{ds} e^{-\varphi(s;t)} \right] t^{-\beta(i,k)} e^{-t} dt
\]

\[
= \int_0^\infty \frac{d^n}{ds^n} [e^{-\varphi(s;t)}] t^{-\beta(i,k)} e^{-t} dt
\]

\[\blacksquare\]

**Lemma 12.** For any integer \( n < \rho, \mu_n < \infty \).

**Proof.** Again, we proceed by induction. \( \mu_0 = 1 \). Now assume that \( \mu_0, \ldots, \mu_{n-1} < \infty \). We begin with the expression for \( \hat{L}^{(n)}(s) \):

\[
\hat{L}^{(n)}(s) = \sum_{i \in I_n} \sum_{k=0}^{l_i} B_{n,i,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(i,k)} e^{-t} dt
\]

Define \( \tilde{I}_n \equiv I_n \setminus (0, \ldots, 0, 1) \). Then pulling out from the sum the term for \( i = (0, \ldots, 0, 1) \), we have

\[
\hat{L}^{(n)}(s) = \frac{\alpha^n \hat{L}^{(n)}(\alpha s)}{\Gamma(1-\alpha)} \int_0^\infty e^{-\varphi(s;t)} t^{-\alpha} e^{-t} dt + \sum_{i \in \tilde{I}_n} \sum_{k=0}^{l_i} B_{n,i,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(i,k)} e^{-t} dt
\]

We can take the limit as \( s \searrow 0 \) of each side. Since \( e^{-\varphi(s;t)} \) is dominated by 1, the limit can be taken inside of each integral, and since \( \lim_{s \searrow 0} \varphi(s;t) = 0 \), we have

\[
\hat{L}^{(n)}(0) = \alpha^n \hat{L}^{(n)}(0) + \sum_{i \in I_n} \sum_{k=0}^{l_i} B_{n,i,k}(0) \Gamma \{1 - \beta(i,k)\}
\]

For each \( i \in \tilde{I}_n \), and for each \( k \), \( B_{n,i,k}(0) \) is proportional to a product of derivatives of \( \hat{L} \), and each of those derivatives is of order less than \( n \). Since all of these are finite, \( \hat{L}^{(n)}(0) < \infty \). \[\blacksquare\]

With this we show that \( \mu_N - (-1)^N \hat{L}^{(N)}(s) \) is regularly varying as \( s \searrow 0 \). First, we have

\[
\mu_N - (-1)^N \hat{L}^{(N)}(s) = (-1)^N \left[ \hat{L}^{(N)}(0) - \hat{L}^{(N)}(s) \right]
\]

\[
= (-1)^N \sum_{i \in I_n} \sum_{k=0}^{l_i} B_{n,i,k}(0) \int_0^\infty t^{-\beta(i,k)} e^{-t} dt - B_{n,i,k}(s) \int_0^\infty e^{-\varphi(s;t)} t^{-\beta(i,k)} e^{-t} dt
\]
We can decompose the object of interest into three terms:

\[
\frac{\mu_N - (-1)^N \hat{L}^{(N)}(s)}{s^{\rho - N}} = A_1(s) + A_2(s) + A_3(s)
\]

where \(A_1\), \(A_2\) and \(A_3\) are defined as

\[
A_1(s) = s^{N-\rho}(-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{l_1} [B_{N,\iota,k}(0) - B_{N,\iota,k}(s)] \int_0^\infty t^{-\beta(\iota,k)} e^{-t} dt
\]

\[
A_2(s) = s^{N-\rho}(-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{l_1} B_{N,\iota,k}(s) \int_0^\infty t^{-\beta(\iota,k)} \left[1 - e^{-\varphi(s,t)}\right] dt
\]

\[
A_3(s) = s^{N-\rho}(-1)^{N+1} \sum_{\iota \in I_N} \sum_{k=0}^{l_1} B_{N,\iota,k}(s) \int_0^\infty t^{-\beta(\iota,k)} \left[1 - e^{-\varphi(s,t)}\right] [1 - e^{-t}] dt
\]

This particular decomposition is useful because it will allow for the use of the Monotone Convergence Theorem in characterizing the limiting behavior of \(A_2\) and \(A_3\).

**Lemma 13.** If \(\rho \notin \mathbb{N}\), \(\lim_{s \searrow 0} A_1(s) = \alpha^\rho \lim_{s \to 0} \frac{\mu_N - (-1)^N \hat{L}^{(N)}(s)}{s^{\rho - N}}\).

**Proof.** As above, we can separate the term for \(\iota = (0,...,0,1)\) from \(I_N\), to write

\[
\lim_{s \searrow 0} A_1(s) = \lim_{s \searrow 0} s^{N-\rho}(-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{l_1} [B_{N,\iota,k}(0) - B_{N,\iota,k}(s)] \Gamma(1 - \beta(\iota,k))
\]

\[
= \lim_{s \searrow 0} (-1)^N \left[\frac{\alpha^N \hat{L}^{(N)}(0)}{\Gamma(1-\alpha)} - \frac{\alpha^N \hat{L}^{(N)}(s)}{\Gamma(1-\alpha)}\right] \Gamma(1 - \alpha)
\]

\[
+ \lim_{s \searrow 0} (-1)^N \sum_{\iota \in I_N} \sum_{k=0}^{l_1} [B_{N,\iota,k}(0) - B_{N,\iota,k}(s)] \frac{1}{s^{\rho - N}} \Gamma(1 - \beta(\iota,k))
\]

The first term is simply \(\alpha^\rho \lim_{s \searrow 0} \frac{\mu_N - (-1)^N \hat{L}^{(N)}(s)}{(as)^{\rho - N}}\). The second term equals zero: After using L'Hospital's rule, the numerator becomes a multinomial of derivatives of \(\hat{L}\) or order no greater than \(N\), (so all of these are finite) while the denominator becomes \((\rho - N)s^{\rho - N-1}\) which goes to infinity.

To characterize the limiting behavior of \(A_2\) and \(A_3\), it will be useful to define

\[
\kappa = \frac{1}{\Gamma(1 - \rho^{-1})} \left[1 - \alpha \|v\| \Gamma(1 - \rho^{-1})\right]
\]

\[
\tilde{\varphi}(s; w) = \varphi \left(s; [1 - \hat{L}(\alpha s)]^\rho \kappa^\rho \alpha^{-\rho} w^{-\rho}\right)
\]
\( \tilde{\varphi} \) is defined this way so that \( \lim_{s \to 0} \tilde{\varphi}(s; w) = w \) (this can be easily verified for each of the three cases: \( \alpha > \nu \), \( \alpha = \nu \), and \( \alpha < \nu \)).

**Lemma 14.** \( \tilde{\varphi}(s; w) \) is non-decreasing in \( s \) in the neighborhood of zero.

**Proof.** Using the definitions of \( \tilde{\varphi} \) and \( \varphi \), we have

\[
\tilde{\varphi}(s; w) = s^{1-\rho\nu}(1 - \alpha) \frac{\kappa^{-\rho\nu} \left[ 1 - \hat{L}(\alpha s) \right]^{-\rho\nu}}{\Gamma(1 - \nu)} w^{\rho\nu} + \left( \frac{\kappa}{\alpha} \right)^{-\rho\alpha} \frac{\rho v}{\Gamma(1 - \alpha)} \left[ 1 - \hat{L}(\alpha s) \right]^{1-\rho\nu} w^{\rho\nu}
\]

We first show that \( \frac{1 - \hat{L}(s)}{s} \) is non-increasing in a neighborhood of zero. Since \( \mu_2 > 0 \),

\[
\lim_{s \searrow 0} \frac{d}{ds} \left( \frac{1 - \hat{L}(s)}{s} \right) = \lim_{s \searrow 0} \frac{-s\hat{L}'(s) - \left[ 1 - \hat{L}(s) \right]}{s^2} = -\hat{L}''(0) < 0 \quad (15)
\]

Next, since \( 1 - \hat{L}(\alpha s) \) is non-decreasing in \( s \), and since both \( 1 - \rho\nu \) and \( 1 - \rho\alpha \) are non-negative, \( \tilde{\varphi}(s; w) \) is nondecreasing in the neighborhood of zero. \( \blacksquare \)

**Lemma 15.** \( \lim_{s \searrow 0} A_2(s) = \kappa^\rho \frac{\rho}{\rho - \rho N} \Gamma(1 - \rho + N) \).

**Proof.** Using the change of variables \( w = \left[ 1 - \hat{L}(\alpha s) \right] \kappa\alpha^{-1} t^{-1/\rho} \), we have

\[
A_2(s) = (-1)^N \sum_{i \in I_N} \sum_{k=0}^{\ell_i} s^{N-\rho\beta(i,k)} B_{N,i,k}(s) \left\{ \frac{1 - \hat{L}(\alpha s)}{\alpha s} \right\}^{\rho-\rho\beta(i,k)} \int_0^\infty \left[ 1 - e^{-\tilde{\varphi}(s; w)} \right] \rho w^{-\rho\beta(i,k)-1} dw
\]

We next take the limit of each side as \( s \) goes to zero. **Lemma 14** and the monotone convergence theorem imply that the limit can be brought inside the integral to yield

\[
\lim_{s \searrow 0} A_2(s) = (-1)^N \sum_{i \in I_N} \sum_{k=0}^{\ell_i} \left( \lim_{s \searrow 0} s^{N-\rho\beta(i,k)} \right) B_{N,i,k}(0) \kappa^{\rho-\rho\beta(i,k)} \int_0^\infty \left[ 1 - e^{-w} \right] \rho w^{-\rho\beta(i,k)-1} dw
\]

Noting that \( N \geq \rho\beta(i,k) \), the term \( \lim_{s \to 0} s^{N-\rho\beta(i,k)} \) is zero unless \( N = \rho\beta(i,k) \). Thus \( \lim_{s \searrow 0} A_2(s) \) can be written as

\[
\lim_{s \searrow 0} A_2(s) = (-1)^N \kappa^{\rho-N} \int_0^\infty \left[ 1 - e^{-w} \right] \rho w^{-(\rho-N)-1} dw \sum_{i \in I_N} \sum_{k=0}^{\ell_i} B_{N,i,k}(0) \mathbb{1}_{N = \rho\beta(i,k)}
\]
The integral is \( \int_0^\infty [1 - e^{-w}] \rho w^{-(\rho - N) - 1} dw = \frac{\rho}{\rho - N} \Gamma(1 - \rho + N) \). To finish the proof, we show that

\[
(-1)^N \sum_{\ell \in I_N} \sum_{k=0}^{\ell_1} B_{N,\ell,k}(0) II_{N=\rho \beta(\ell,k)} = \kappa^N
\]  

To see this note first that \( N = \rho \beta(\ell, k) \) requires \( \ell = (N, 0, \ldots, 0) \). If \( \alpha > v \), \( N = \rho \beta(\ell, k) \) also requires \( k = 0 \), whereas if \( \alpha < v \), it requires \( k = N \). If \( \alpha = v \), \( N = \rho \beta(\ell, k) \) for each \( k \in \{0, \ldots, N\} \). For each of these three cases, one can compute each non-zero term in the sum and verify equation (16).

Lemma 16. \( \lim_{s \searrow 0} A_3(s) = 0 \)

**Proof.** The strategy is the same as in the previous lemma. Using the same change of variables \( w = \left[1 - \hat{\mathcal{L}}(\alpha s)\right] \kappa^{-1} t^{-1/\rho} \), we have

\[
A_3(s) = (-1)^N \sum_{\ell \in I_N} \sum_{k=0}^{\ell_1} s^{N-\rho \beta(\ell,k)} B_{N,\ell,k}(s) \left\{ \frac{1 - \hat{\mathcal{L}}(\alpha s)}{\alpha s} \right\}^{\rho-\rho \beta(\ell,k)} \times \int_0^\infty \left[1 - e^{-\hat{\varphi}(s,w)}\right] \left[1 - e^{-[1 - \hat{\mathcal{L}}(\alpha s)]^\rho (\kappa/\alpha)^{\rho w - \rho}}\right] \rho w^{\rho - \rho \beta(\ell,k) - 1} dw
\]

We can take a limit of each side. Since both \( \hat{\varphi}(s; w) \) and \( \left[1 - \hat{\mathcal{L}}(\alpha s)\right]^\rho (\kappa/\alpha)^{\rho w - \rho} \) are non-decreasing in \( s \) in the neighborhood of \( s = 0 \), we can use the monotone convergence theorem to bring the limit inside the integral. Thus we have

\[
\lim_{s \searrow 0} A_3(s) = (-1)^N \sum_{\ell \in I_N} \sum_{k=0}^{\ell_1} \left( \lim_{s \searrow 0} s^{N-\rho \beta(\ell,k)} \right) B_{N,\ell,k}(0) \kappa^{\rho-\rho \beta(\ell,k)} \times \int_0^\infty \lim_{s \searrow 0} \left[1 - e^{-\hat{\varphi}(s,w)}\right] \left[1 - e^{-[1 - \hat{\mathcal{L}}(\alpha s)]^\rho (\kappa/\alpha)^{\rho w - \rho}}\right] \rho w^{\rho - \rho \beta(\ell,k) - 1} dw
\]

Since the limit of the integrand is zero for each integral, we have \( \lim_{s \searrow 0} A_3(s) = 0 \). □

We finally come to the main result.

**Proposition (6b).** If \( \rho \not\in \mathbb{N} \), \( 1 - \mathcal{L}(l) \sim \frac{\kappa^\rho}{1 - \alpha l^{1-\rho}} \)

**Proof.** The previous lemmas imply that

\[
\lim_{s \searrow 0} \frac{\mu_N - (-1)^N \hat{\mathcal{L}}^{(N)}(s)}{s^{\rho - N}} = \alpha^\rho \lim_{s \searrow 0} \frac{\mu_N - (-1)^N \hat{\mathcal{L}}^{(N)}(s)}{s^{\rho - N}} + \kappa^\rho \frac{\rho}{\rho - N} \Gamma(1 - \rho + N) + 0
\]

63
or
\[
\lim_{s \searrow 0} \frac{\mu_N - (-1)^N \hat{L}(N)(s)}{s^{\rho-N}} = \frac{\kappa^\rho}{1 - \alpha^\rho} \frac{\rho}{\Gamma(1 - \rho + N)}
\]

By Theorem A in Bingham and Doney (1974) we therefore have that
\[
\lim_{l \to \infty} \frac{1 - \mathcal{L}(l)}{l^{-\rho}} = \frac{\kappa^\rho}{1 - \alpha^\rho} \left[ (-1)^N \frac{\Gamma(\rho - N)}{\Gamma(\rho)} \frac{\Gamma(1 - \rho + N)}{\Gamma(1 - \rho)} \right]
\]

Since \(\Gamma(\rho) = \Gamma(\rho - N) \prod_{k=1}^{N} (\rho - k)\) and \((-1)^N \Gamma(1 - \rho + N) = \Gamma(1 - \rho) \prod_{k=1}^{N} (-1)(1 - \rho + N - k) = \Gamma(1 - \rho) \prod_{k=1}^{N} (\rho - k)\), the term in brackets equals unity, completing the proof.

Next we turn to the tail behavior of the conditional size distribution, \(\mathcal{L}(\cdot | q)\).

**Proposition (6c).** If \(\rho \not\in \mathbb{N}\), \(1 - \mathcal{L}(l | q) \sim \frac{m_\rho q^\alpha}{b} \alpha^\rho \left[ 1 - \mathcal{L}(l) \right] \)

**Proof.** Following the logic of Lemma 10, the transform of \(\mathcal{L}(\cdot | q)\) can be written as \(\hat{L}(s | q) = e^{-\varphi(s, \theta q^{-\zeta})}\), with derivatives
\[
\hat{L}'(s | q) = e^{-\varphi(s, \theta q^{-\zeta})} \sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} B_{n,i,k}(s) \left( \theta q^{-\zeta} \right)^{-\beta(i,k)}
\]

For \(k \leq n < \rho\), \(B_{n,i,k}(0)\) is finite, so \(\mu_n(q) = \hat{L}'(n)(0 | q) < \infty\) for each \(n < \rho\). Then, using \(t(q) = \theta q^{-\zeta}\), we have
\[
\lim_{s \searrow 0} \frac{\hat{L}(N)(0 | q) - \hat{L}(N)(s | q)}{s^{\rho-N}} = \lim_{s \searrow 0} \frac{\sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} B_{n,i,k}(0) t(q)^{-\beta(i,k)} - e^{-\varphi(s, t(q))} \sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} B_{n,i,k}(s) t(q)^{-\beta(i,k)}}{s^{\rho-N}}
\]
\[
= \lim_{s \searrow 0} \sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} B_{n,i,k}(0) t(q)^{-\beta(i,k)} - e^{-\varphi(s, t(q))} \sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} B_{n,i,k}(s) t(q)^{-\beta(i,k)}
\]
\[
+ \lim_{s \searrow 0} \frac{1 - e^{-\varphi(s, t(q))}}{s^{\rho-N}} \sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} B_{n,i,k}(s) t(q)^{-\beta(i,k)}
\]

Using the logic of Lemma 13, the first term is
\[
\lim_{s \searrow 0} \sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} B_{n,i,k}(0) - B_{n,i,k}(s) \frac{t^{-\beta(i,k)}}{s^{\rho-N}} = \lim_{s \searrow 0} \frac{\alpha^N \hat{L}(N)(0)}{\Gamma(1 - \alpha)} - \frac{\alpha^N \hat{L}(N)(s)}{\Gamma(1 - \alpha)} t^{-\alpha}
\]
\[
+ \lim_{l \searrow 0} \sum_{i \in I_n} \sum_{k=0}^{\lfloor \frac{1}{\rho} \rfloor} \left[ B_{n,i,k}(0) t^{-\beta(i,k)} - B_{n,i,k}(s) \right] \frac{t^{-\beta(i,k)}}{s^{\rho-N}}
\]
\[
= \frac{\alpha^\rho t^{-\alpha}}{\Gamma(1 - \alpha)} \lim_{s \searrow 0} \frac{\hat{L}(N)(0) - \hat{L}(N)(s)}{s^{\rho-N}}
\]
The second term is zero because, using L’Hospital gives
\[
\lim_{s \downarrow 0} \frac{1 - e^{-\varphi(s,t)}}{s^{\rho - N}} = \lim_{s \downarrow 0} \frac{\varphi^{(1)}(s,t) e^{-\varphi(s,t)}}{(\rho - N) s^{\rho - N - 1}} = 0
\]

We thus have
\[
\lim_{s \downarrow 0} \frac{\hat{L}^{(N)}(0|q) - \hat{L}^{(N)}(s|q)}{s^{\rho - N}} = \frac{\alpha (\theta q^{-\zeta})^{-\alpha}}{\Gamma(1 - \alpha)} \lim_{s \downarrow 0} \frac{\hat{L}^{(N)}(0) - \hat{L}^{(N)}(s)}{s^{\rho - N}}
\]

The result follows from Theorem A in Bingham and Doney (1974) and noting that \( \frac{\alpha}{\eta} = \frac{\theta - \alpha}{\Gamma(1 - \alpha)} \).

C.3 Correlations

Proposition (7). With the functional forms, (i) \( \text{Corr}(\log z, \log q_s) = -\left[1 + \delta(\alpha)^{-1}\right]^{-1/2} \), (ii) \( \text{Corr}(\log q_b, \log q_s) = 0 \), (iii) \( \text{Corr}(\log z, \log q_b) = \left[1 + \delta(\alpha) \left(1 + 2\delta(\alpha)\right)\right]^{-1/2} \), where \( \delta(\alpha) = \frac{6\alpha^2}{\pi^2} \left(\frac{\Gamma''(1 - \alpha)}{\Gamma(1 - \alpha)} - \frac{\Gamma'(1 - \alpha)^2}{2\Gamma(1 - \alpha)^2}\right) \) is positive and increasing for \( \alpha \in (0, 1) \).

Proof. We characterize the joint distribution of match-specific productivity and supplier’s efficiency as follows: Consider an entrepreneur with efficiency \( q_s \). If that entrepreneur has a downstream technique with productivity \( z \), the associated buyer selects that technique with probability \( \tilde{F}(z q_s^{\alpha}) \) (defined in Proposition 3). Since the average number of downstream techniques is \( M \), we can integrate over the distributions of productivity draws and of potential suppliers’ efficiencies to get that among techniques that are actually used, the fraction with productivity no greater than \( \hat{z} \) and that are associated with suppliers with efficiency no greater than \( \hat{q}_s \) is

\[
\int_0^{\hat{q}_s} \int_{\hat{z}}^\infty M \tilde{F}(z q_s^{\alpha}) dH(z) dF(q_s)
\]

This joint CDF can be used to characterize various cross-sectional moments in the model. With the functional forms, we can use \( M z_0^\zeta = m \to \frac{\theta^{1-\alpha}}{\Gamma(1 - \alpha)} \) and \( \tilde{F} \to F \), so that the joint density is

\[
\frac{\theta^{1-\alpha}}{\Gamma(1 - \alpha)} e^{-\theta z^{-\zeta} q_s^{-\alpha \zeta}} \zeta z^{-\zeta - 1} e^{-\theta q_s^{-\zeta}} \zeta q_s^{-\zeta - 1}
\]

Next, for any \( k_1, k_2, k_3 \), we have

\[
\mathbb{E} \left[ (\log z)^{k_1} (\log q_s)^{k_2} (\log q_s^{\alpha})^{k_3} \right] = \int_0^\infty \int_0^\infty \left\{ (\log z)^{k_1} (\log q_s)^{k_2} (\log q_s^{\alpha})^{k_3} \times \frac{\theta^{1-\alpha}}{\Gamma(1 - \alpha)} e^{-\theta z^{-\zeta} q_s^{-\alpha \zeta}} \zeta z^{-\zeta - 1} e^{-\theta q_s^{-\zeta}} \zeta q_s^{-\zeta - 1} \right\} \ dz dq_s
\]

65
Using the change of variables $x = \theta q^{-\zeta}$, $u = \theta (zq)^{\alpha} - \zeta$, the right hand side becomes

$$
\frac{\zeta^{-(k_1+k_2+k_3)}}{\Gamma(1-\alpha)} \int_0^\infty \int_0^\infty \left\{ [\log \theta^{1-\alpha} + \alpha \log x - \log u]^k_1 [\log \theta - \log x]^k_2 \right\} dudx
$$

This formula can be used for various values of $\{k_1, k_2, k_3\}$ to evaluate each first and second moment.

The expressions for the correlations follow from this, the fact that for any $t$ and positive integer $n$,

$$
\int_0^\infty (\log x)^n x^{t-1} e^{-x} dx = \frac{d^n}{dt^n} \int_0^\infty x^{t-1} e^{-x} dx = \Gamma^{(n)}(t),
$$

and noting that $\Gamma^{(2)}(1) - \Gamma^{(1)}(1)^2 = \pi^2/6$.

## D The Number of Supply Chains

For completeness, we show the derivation of several results from the theory of branching processes that are used in this paper (see for example Athreya and Ney (1972)).

Let $B_N(j) \equiv |C_N(j)|$ be the number of distinct chains of length $N$. Let $p(k)$ be the probability that a firm has exactly $k$ techniques, in this case equal to $e^{-M} M^k / k!$, and let $P_N(l, k)$ be the probability that, in total, $l$ different firms have among them $k$ supply chains of length $N$ (i.e. $\sum_{i=1}^l B_N(j_i) = k$). Note that $P_N(1, k)$ is the probability a firm has exactly $k$ supply chains of length $N$ (i.e., the probability that $B_N = k$).

Define $\varphi(x) = \sum_{k=0}^\infty p(k)x^k$ to be the probability generating function for the random variable $B_1$. In this case $\varphi(x) = e^{-M(1-x)}$. Also, for each $N$, let $\varphi_N(\cdot)$ be the probability generating function associated with $B_N$. If $\varphi^{(N)}$ is the $N$-fold composition of $\varphi$ then we have the convenient result:

**Lemma 17.** $\varphi_N(x) = \varphi^{(N)}(x)$

**Proof.** We proceed by induction. By definition, the statement is true for $N = 1$. Noting that $\sum_{k=0}^\infty P_1(l, k)x^k = \varphi(x)^l$, we have

$$
\varphi_{N+1}(x) = \sum_{l=0}^\infty P_{N+1}(1, l)x^l = \sum_{l=0}^\infty \sum_{k=0}^\infty P_N(1, k)P_1(k, l)x^l = \sum_{k=0}^\infty P_N(1, k) \sum_{l=0}^\infty P_1(k, l)x^l
$$

$$
= \sum_{k=0}^\infty P_N(1, k)\varphi(x)^k = \varphi_N(\varphi(x))
$$

We immediately have the following:

**Claim 1.** For any $x$, $\mathbb{E}[x^{B_N}] = \varphi^{(N)}(x)$. 

66
Proof. $\mathbb{E}[x^B_N] = \sum_{k=0}^{\infty} P_N(1,k)x^k = \varphi_N(x) = \varphi^{(N)}(x)$

We next study the probability that a firm has no chains that continue indefinitely.

**Claim 2.** The probability that a single firm has no chains that continue indefinitely is the smallest root, $\rho$, of $y = \varphi(y)$.

**Proof.** The probability that a firm has no chains greater than length $N$ is $P_N(1,0)$, or $\varphi_N(0)$. Then the probability that a single firm has no chains that continue indefinitely is $\lim_{N \to \infty} \varphi^{(N)}(0)$. Next note that that $\varphi$ is increasing and convex, $\varphi(1) = 1$, and $\varphi(0) \geq 0$. This implies that in the range $[0,1]$ the equation $\varphi(y) = y$ has a either a unique root at $y = 1$ or two roots, $y = 1$ and a second in $(0,1)$.

Let $\rho$ be the smallest root. Noting that for $y \in [0,\rho)$, $y < \varphi(y) < \rho$, while for $y \in (\rho,1)$ (if such $y$ exist), $\rho < \varphi(y) < y$. Together these imply that if $y \in [0,1)$, the sequence $\{\varphi^{(N)}(y)\}$ is monotone and bounded, and therefore has a limit. We have $\varphi^{(N+1)}(0) = \varphi(\varphi^{(N)}(0))$. Taking limits of both sides (and noting that $\varphi$ is continuous) gives $\lim_{N \to \infty} \varphi^{(N+1)}(0) = \varphi(\lim_{N \to \infty} \varphi^{(N)}(0))$. Therefore the limit is a root of $y = \varphi(y)$, and therefore must be $\rho$. In other words, $\lim_{N \to \infty} \varphi^{(N)}(0) = \rho$.

**Claim 3.** If $M \leq 1$ then with probability 1 the firm has no chains that continue indefinitely. If $M > 1$ then there is a strictly positive probability the firm has a chain that continues indefinitely.

**Proof.** In this case, we have $\varphi(x) = e^{-M(1-x)}$. If $M \leq 1$ then the smallest root of $y = \varphi(y)$ is $y = 1$. If $M > 1$ the smallest root is strictly less than 1.

### E Dynamics

**Claim 4.** The joint distribution over efficiencies at $t$ and $\hat{t}$ is

$$F_{t,\hat{t}}(q,\hat{q}) = e^{-M_{\hat{t}} \int_0^{\infty} \left[1 - F_{\hat{t},\hat{t}}\left(\frac{q}{z}, \left(\frac{\hat{q}}{\hat{z}}\right)^{1/\alpha}\right)\right]dH(z)} e^{-\left(M_{t}-M_{\hat{t}}\right) \int_0^{\infty} \left[1 - F_t\left(\frac{q}{z}\right)^{1/\alpha}\right]dH(z)}$$

**Proof.** Let $G_{t,\hat{t}}(q,\hat{q})$ be the probability that a single technique that exists at $t$ delivers efficiency greater than $q$ at $t$ or greater than $\hat{q}$ at $\hat{t}$. This can be written as

$$G_{t,\hat{t}}(q,\hat{q}) = \int_{\mathbb{R}_+} F_{t,\hat{t}}\left(\left(\frac{q}{z}\right)^{1/\alpha}, \left(\frac{\hat{q}}{\hat{z}}\right)^{1/\alpha}\right) dH(z)$$
To interpret this, $F_{t, \hat{t}} \left( \left( \frac{q}{z} \right)^{\frac{1}{\alpha}}, \left( \frac{\hat{q}}{z} \right)^{\frac{1}{\alpha}} \right)$ is the fraction of suppliers with efficiency low enough at $t$ and $\hat{t}$ so that, in combination with efficiency $z$, the technique delivers efficiency no greater than $q$ at $t$ and no greater than $\hat{q}$ at $\hat{t}$. The number of techniques discovered by each firm before $t$ follows a Poisson distribution with mean $M_t$, so the probability that all such techniques deliver efficiency no greater than $q$ at $t$ and no greater than $\hat{q}$ at $\hat{t}$ is $e^{-M_t[1-G_{t, \hat{t}}(q, \hat{q})]}$.

Second, a firm discovers techniques between $t$ and $\hat{t}$. The probability that a single one of these techniques delivers efficiency no greater than $\hat{q}$ at $\hat{t}$ is $G_t(\hat{q}) = \int_{\hat{q}}^{\infty} F_H \left( \left( \frac{\hat{q}}{z} \right)^{1/\alpha} \right) dz$. The number of techniques discovered between $t$ and $\hat{t}$ follows a Poisson distribution with mean $M_t - M_t$, so the probability that the firm has discovered no such techniques is $e^{-(M_t-M_t)[1-G_t(\hat{q})]}$.

In this section, the subscripts $t$ and $\hat{t}$ will be omitted except when necessary for clarity.

**Claim 5.** With the functional form, $F_{t, \hat{t}}(q, \hat{q}) = F_t(q) \tau(q/\hat{q})$ where $\tau$ satisfies

$$\tau(x) = \tau \left( x^{1/\alpha} \right)^{\alpha} + \left( \frac{\theta_{\hat{t}}}{\theta_t} - \left( \frac{\theta_{\hat{t}}}{\theta_t} \right)^{\alpha} \right) x^{-\alpha} \quad (17)$$

**Proof.** With the functional forms, we have for $k > 0$

$$F_{t, \hat{t}}(kq, k\hat{q}) = e^{-mk} f_0 \left[ 1 - F_{t, \hat{t}} \left( \left( \frac{kq}{\hat{q}} \right)^{1/\alpha}, \left( \frac{k\hat{q}}{\hat{q}} \right)^{1/\alpha} \right) \right] \zeta^{-\alpha-1}d\zeta - e^{-m(k-1)} f_0 \left[ 1 - F_{t, \hat{t}} \left( \left( \frac{kq}{\hat{q}} \right)^{1/\alpha}, \left( \frac{k\hat{q}}{\hat{q}} \right)^{1/\alpha} \right) \right] \zeta^{-\alpha-1}d\zeta

= e^{-mk^{-\alpha}} f_0 \left[ 1 - F_{t, \hat{t}} \left( \left( \frac{q}{z} \right)^{1/\alpha}, \left( \frac{\hat{q}}{z} \right)^{1/\alpha} \right) \right] \zeta^{-\alpha-1}d\zeta - (m_k) f_0 \left[ 1 - F_{t, \hat{t}} \left( \left( \frac{q}{z} \right)^{1/\alpha}, \left( \frac{\hat{q}}{z} \right)^{1/\alpha} \right) \right] \zeta^{-\alpha-1}d\zeta

= F_{t, \hat{t}}(q, \hat{q}) k^{-\alpha}

Define $\tau(x) \equiv -\frac{1}{\theta_t} \log F_{t, \hat{t}}(1, x)$ so that $F_{t, \hat{t}}(q, \hat{q}) = e^{-\theta_t q^{-\alpha} \tau(q/\hat{q})} = F_t(q) \tau(q/\hat{q})$. Then using the definition of $\tau$, we have

$$\tau(x) = \frac{m_t}{\theta_t} \int_0^\infty \left[ 1 - F_{t, \hat{t}} \left( \left( \frac{x}{z} \right)^{1/\alpha}, \left( \frac{x}{z} \right)^{1/\alpha} \right) \right] \zeta^{-\alpha-1}d\zeta + \frac{m_t - m_{\hat{t}}}{\theta_t} \int_0^\infty \left[ 1 - F_{t, \hat{t}} \left( \left( \frac{x}{z} \right)^{1/\alpha}, \left( \frac{x}{z} \right)^{1/\alpha} \right) \right] \zeta^{-\alpha-1}d\zeta

= \frac{m_t}{\theta_t} \int_0^\infty \left[ 1 - e^{-\theta_t \zeta^\frac{1}{\alpha}} \right] \zeta^{-\alpha-1}d\zeta + \frac{m_t - m_{\hat{t}}}{\theta_t} \int_0^\infty \left[ 1 - e^{-\theta_t \zeta^\frac{1}{\alpha}} \right] \zeta^{-\alpha-1}d\zeta

= \frac{m_t}{\theta_t} \left( \theta_t \frac{\alpha}{\Gamma(1-\alpha)} \right)^\alpha \Gamma(1-\alpha) + \frac{m_t - m_{\hat{t}}}{\theta_t} \left( \left( \frac{x}{z} \right)^{-\alpha} \right)^\alpha \Gamma(1-\alpha)

= \left( \frac{x}{z} \right)^{-\alpha} \left( \frac{\theta_t}{\theta_t} - \left( \frac{\theta_t}{\theta_t} \right)^{\alpha} \right) x^{-\alpha}
E.1 Increase in Efficiency of a Technique

How much does the efficiency of an individual single technique improve between \( t \) and \( \hat{t} \)? Among techniques that deliver efficiency \( q \) at \( t \), let \( I_{t,\hat{t}}(x|q) \) be the fraction that deliver efficiency no greater than \( xq \) at \( \hat{t} \).

Claim 6.

\[
I_{t,\hat{t}}(x|q) = \frac{\int_{0}^{(q/z_0)^{1/\alpha}} \frac{1}{q_s} H' \left( \frac{q}{q_s} \right) \frac{\partial F_{t,\hat{t}}(q_s, q_s x^{1/\alpha})}{\partial q} dq_s}{\int_{0}^{(q/z_0)^{1/\alpha}} \frac{1}{q_s} H' \left( \frac{q}{q_s} \right) dF_{t}(q_s)}
\]

(18)

Proof.

The probability that the efficiency of a single technique increases by the proportion \( x \) between \( t \) and \( \hat{t} \) is the same as the probability that the associated supplier’s efficiency increases by the proportion \( x^{1/\alpha} \).

For a supplier with efficiency \( q_s \) at \( t \), the probability that a single downstream technique delivers efficiency no greater than \( q \) at \( t \) is \( H(q/q_s^{\alpha}) \), with density \( \frac{1}{q_s^{\alpha}} H' \left( \frac{q}{q_s} \right) \). Bayes rule implies that the conditional density of suppliers’ efficiency among techniques that deliver efficiency \( q \) at \( t \) is

\[
\frac{1}{q_s^{\alpha}} H' \left( \frac{q}{q_s} \right) \frac{F_{t}'(q_s)}{\int_{0}^{(q/z_0)^{1/\alpha}} \frac{1}{q_s} H' \left( \frac{q}{q_s} \right) dF_{t}(q_s)}
\]

Among suppliers with efficiency \( q_s \) at \( t \), the fraction with efficiency no greater than \( q_s x^{1/\alpha} \) at \( \hat{t} \) is \( \frac{1}{F_{t}'(q_s)} \frac{\partial F_{t,\hat{t}}(q_s, q_s x^{1/\alpha})}{\partial q} \), where \( \frac{\partial F_{t,\hat{t}}(\cdot, \cdot)}{\partial q} \) is the partial derivative with respect to its first argument.

To find, \( I \), we integrate the probability that the supplier’s efficiency improved by no more than \( x^{1/\alpha} \) over the possible values of the supplier’s efficiency.

Claim 7. With the functional forms, \( I(x|q) = \tau(x) + \frac{1}{\zeta} \tau'(x) x \).

Proof. First, we use the functional forms and take the limit as \( z_0 \to 0 \) to get

\[
\frac{\int_{0}^{q/z_0^{1/\alpha}} \frac{1}{q_s} H' \left( \frac{q}{q_s} \right) dF_{t}(q_s)}{\int_{0}^{q/z_0^{1/\alpha}} \frac{1}{q_s} H' \left( \frac{q}{q_s} \right) dF_{t}(q_s)} \to \int_{0}^{\infty} \frac{q_s^{\alpha\zeta}}{q_s^{1-\alpha} q_s^{\zeta-1} e^{-\theta q_s^{\zeta}}} \frac{q_s^{\alpha\zeta}}{\theta e^{\alpha} \Gamma(1-\alpha)}
\]

Note that this expression is independent of the efficiency delivered by the technique.

Second, we have that with the functional forms we have \( F_{t,\hat{t}}(q, \hat{q}) = e^{-\theta q^{1-\zeta-\zeta} \tau(\hat{q}/q)} \), which means the derivative with respect to its first argument is \( \frac{\partial F_{t,\hat{t}}(q, \hat{q})}{\partial q} = \left[ \tau \left( \frac{\hat{q}}{q} \right) + \frac{1}{\zeta} \tau' \left( \frac{\hat{q}}{q} \right) \right] \theta q^{-\zeta-1} e^{-\theta q^{1-\zeta-\zeta} \tau(\hat{q}/q)} \). Thus we have
\[ \frac{\partial F_{t,i}(q_s, q_s x^{1/\alpha})}{\partial q} = \left[ \tau(x^{1/\alpha}) + \frac{1}{\zeta} \tau'(x^{1/\alpha}) x^{1/\alpha} \right] \theta_t \zeta q_s^{-\zeta - 1} e^{-\theta_t q_s^{-\zeta}} \tau(x^{1/\alpha}) \]

We can simplify the term in brackets by using \( \tau(x^{1/\alpha}) = \tau(x) - \left( \frac{\theta_t}{\alpha} - \left( \frac{\theta_t}{\alpha} \right)^{\alpha} \right) x^{-\zeta} \) and its derivative, \( \tau'(x^{1/\alpha}) x^{1/\alpha - 1} = \tau'(x) - \left( \frac{\theta_t}{\alpha} - \left( \frac{\theta_t}{\alpha} \right)^{\alpha} \right) (-\zeta) x^{-\zeta - 1} \), so this becomes

\[ \frac{\partial F_{t,i}(q_s, q_s x^{1/\alpha})}{\partial q} = \left[ \tau(x) + \frac{1}{\zeta} \tau'(x) x \right] \tau(x^{1/\alpha})^{1-\alpha} \theta_t \zeta q_s^{-\zeta - 1} e^{-\theta_t q_s^{-\zeta}} \tau(x^{1/\alpha}) \]

Finally, equation 1 can be written as

\[ I(x|q) \rightarrow \int_0^\infty \frac{q_s^{\alpha \zeta}}{\theta_t^2 \Gamma(1-\alpha)} \left[ \tau(x) + \frac{1}{\zeta} \tau'(x) x \right] \tau(x^{1/\alpha})^{1-\alpha} \theta_t \zeta q_s^{-\zeta - 1} e^{-\theta_t q_s^{-\zeta}} \tau(x^{1/\alpha}) dq_s \]

\[ = \tau(x) + \frac{1}{\zeta} \tau'(x) x \]

### E.2 Conditional Distributions

This section derives expressions for the conditional distributions, \( R^{\text{same}} \), \( R^{\text{unused}} \), and \( R^{\text{new}} \).

**Claim 8.** The conditional distributions satisfy

1. \( R^{\text{same}}(\hat{q}|q) = I\left( \frac{\hat{q}}{q} \right) \)
2. \( R^{\text{unused}}(\hat{q}|q) = e^{-M_t \int_0^\infty \left[ 1-I\left( \frac{\hat{q}}{\tilde{q}} \right) \right] dG_t(\tilde{q})} \)
3. \( R^{\text{new}}(\hat{q}) = e^{-(M_t-M_t)\left[ 1-G_t(\hat{q}) \right]} \)

**Proof.** That \( R^{\text{same}}(\hat{q}|q) = I\left( \frac{\hat{q}}{q} \right) \) follows from the definition of \( I \). The expression for \( R^{\text{new}}(\hat{q}) \) can be derived in the same way as the derivation of as the cross sectional distribution. Turning to \( R^{\text{unused}} \), consider a single technique with efficiency no greater than \( q \) at \( t \). What is the probability that it delivers efficiency no greater than \( \hat{q} \) at \( \hat{t} \)? The density of efficiency at \( t \) among such techniques is \( \frac{G_t(\hat{q})}{G_t(q)} \), so the probability that the technique delivers efficiency no greater than \( \hat{q} \) at \( \hat{t} \) is \( \int_0^\infty \frac{I(\hat{q}/\tilde{q}) G_t(\tilde{q})}{G_t(q)} dG_t(\tilde{q}) \).

Among firms with efficiency \( \tilde{q} \) at \( t \), the fraction with \( n \) techniques at \( t \) can be computed using
Bayes rule:

\[
\text{Pr}(n \text{ techniques given efficiency } q) = \frac{e^{-M_t M_n^n} n G_t(q)^{n-1} G'_t(q)}{\sum_{k=1}^\infty e^{-M_t M_k^n} k G_t(q)^{k-1} G'_t(q)} \cdot \frac{e^{-M_t M_n^{n-1}} G_t(q)^{n-1}}{F_i(q)}
\]

The numerator is the product the probability that a firm discovered \( n \) techniques and the probability that most efficient of these delivers efficiency \( q \). The denominator is the sum of this quantity over the possible number of techniques.

For a firm with \( n \) techniques and efficiency \( q \) at \( t \), the probability that the \( n - 1 \) that are not used at \( t \) all deliver efficiency no greater than \( \tilde{q} \) at \( \hat{t} \) is \( \left( \frac{\int_0^\tilde{q} I(\tilde{q}/\tilde{q}) dG_t(\tilde{q})}{G_t(\tilde{q})} \right)^{n-1} \). Therefore among all firms with efficiency \( q \) at \( t \), the probability that no unused techniques deliver efficiency greater than \( \tilde{q} \) at \( t \) is

\[
R^{\text{unused}}(\tilde{q}|q) = \sum_{n=1}^\infty \frac{e^{-M_t M_n^{n-1}} G_t(q)^{n-1}}{(n-1)!} \left( \frac{\int_0^{\tilde{q}} I(\tilde{q}/\tilde{q}) dG_t(\tilde{q})}{G_t(\tilde{q})} \right)^{n-1} F_i(q)
\]

\[
= \sum_{n=1}^\infty \frac{e^{-M_t M_n^{n-1}}}{(n-1)!} \left( \int_0^{\tilde{q}} I(\tilde{q}/\tilde{q}) dG_t(\tilde{q}) \right)^{n-1} F_i(q)
\]

\[
= e^{-M_t \left[1-\int_0^{\tilde{q}} I(\tilde{q}/\tilde{q}) dG_t(\tilde{q})\right]} F_i(q)
\]

\[
= e^{-M_t \int_0^{\tilde{q}} \left[1-I(\tilde{q}/\tilde{q})\right] dG_t(\tilde{q})}
\]

Claim 9. With the functional forms, these are given by

1. \( R^{\text{same}}(\tilde{q}|q) = \tau \left( \frac{\tilde{q}}{q} \right) + \frac{1}{\tilde{q}} \tau' \left( \frac{\tilde{q}}{q} \right) \frac{\tilde{q}}{q} \)

2. \( R^{\text{unused}}(\tilde{q}|q) = e^{-\theta t \tilde{q}^{-\xi} \left[ \frac{m_t}{\tilde{m}} - 1 \right]} - \theta t q^{-\xi} [\tau(\tilde{q}/q) - 1] \)

3. \( R^{\text{new}}(\tilde{q}) = e^{-\theta t \tilde{q}^{-\xi} \left[ 1 - \frac{m_t}{\tilde{m}} \right]} \)

Proof. The expression for \( R^{\text{same}} \) follows directly from the expression for \( I \). For \( R^{\text{new}} \) we have

\[
R^{\text{new}}(\tilde{q}|q) = e^{-(M_t - M_i) \left[1-G_t(\tilde{q})\right]} = e^{-M_i \left[1-G_t(\tilde{q})\right] \frac{M_t - M_i}{M_i}} \to e^{-\theta t \tilde{q}^{-\xi} \left[1 - \frac{m_t}{\tilde{m}}\right]}
\]

To find \( R^{\text{unused}} \), we first have that with the functional forms, \( \lim_{x_0 \to 0} M_t G'_t(\tilde{q}) = \theta t \tilde{q}^{-\xi} \). Thus
the integral

\[ M_t \int_0^q \left[ 1 - I(\hat{q}/\hat{q}|q) \right] dG_t(\hat{q}) \rightarrow \int_0^q \left[ 1 - I(\hat{q}/\hat{q}) \right] \theta_t \zeta \hat{q}^{-\zeta-1} d\hat{q} \]

\[ = \theta_t \hat{q}^{-\zeta} \int_{\hat{q}/q}^\infty \left[ 1 - I(u) \right] \zeta u^{\zeta-1} d\hat{q} \]

We can substitute in the expression for \( I \) to get

\[ \int_{\hat{q}/q}^\infty \left[ 1 - I(u) \right] \zeta u^{\zeta-1} d\hat{q} = \int_{\hat{q}/q}^\infty \left[ 1 - \tau(u) \right] \zeta u^{\zeta-1} - \tau'(u) u^\zeta d\hat{q} \]

\[ = \left[ 1 - \tau(u) \right] u^{\zeta} \left|_{\hat{q}/q}^\infty \right. \]

To find \( \lim_{u \to \infty} \frac{1 - \tau(u)}{u^\zeta} \), note that we can rewrite equation (17) as

\[ \frac{\tau(u) - 1}{u^{-\zeta}} = \frac{\tau(u/\alpha)}{\left[ \tau(u/\alpha) - 1 \right]^{\alpha}} \left[ \frac{\tau(u/\alpha) - 1}{(u/\alpha)^{-\zeta}} \right]^{\alpha} + \left( \frac{\theta_t}{\alpha} \right)^{\alpha} - \left( \frac{\theta_t}{\alpha} \right) \]

Since \( \lim_{u \to \infty} \frac{\tau(u/\alpha) - 1}{\left[ \tau(u/\alpha) - 1 \right]^{\alpha}} = 0 \), we have \( \lim_{u \to \infty} \frac{1 - \tau(u)}{u^\zeta} = \left( \frac{\theta_t}{\alpha} \right)^{\alpha} - \left( \frac{\theta_t}{\alpha} \right) \). Putting these pieces together, we have

\[ R(\hat{q} | q) \rightarrow e^{-\theta_t \hat{q}^{-\zeta} \left( \left\{ \frac{\theta_t}{\alpha} \right\}^{\alpha} - \frac{\theta_t}{\alpha} - [1 - \tau(\hat{q}/q)](\hat{q}/q)^\zeta \right) = e^{-\left\{ \frac{\theta_t}{\alpha} \right\}^{\alpha} - \theta_t \hat{q}^{-\zeta} - \theta_t q^{-\zeta} [\tau(\hat{q}/q) - 1] } \]

E.3 Surplus

The average surplus among firms with efficiency \( q \) is

\[ S(q) = \frac{(q/Q)^{\zeta-1}}{\zeta-1} + M \int_0^\infty \int_0^\infty [S(\max\{\hat{q}, zq^\alpha\}) - S(\hat{q})] d\hat{F}(\hat{q}) dH(z) \]

Claim 10. Using the functional forms, \( \lim_{z \to 0} S(q) = \frac{(q/Q)^{\zeta-1}}{\zeta-1} + \frac{mq^{\alpha}}{\theta} \frac{1 - \frac{1}{\zeta - 1}}{1 - \alpha} \frac{1}{\zeta - 1} \]

Proof. Consider the surplus created by all downstream techniques. Using the functional forms,
this can be written as

\[ M \int_{z_0}^{\infty} \int_{0}^{zq^\alpha} [S(zq^\alpha) - S(\tilde{q})] \, d\tilde{F}(\tilde{q}) \, dH(z) = mz_0^{-\zeta} \int_{z_0}^{\infty} \int_{0}^{zq^\alpha} [S(zq^\alpha) - S(\tilde{q})] \, d\tilde{F}(\tilde{q}) z_0^{-\zeta} z^{-\zeta-1} \, dz \]

\[ = mq\zeta\alpha \int_{z_0q^\alpha}^{\infty} \int_{0}^{w} [S(w) - S(\tilde{q})] \, d\tilde{F}(\tilde{q}) w^{-\zeta-1} \, dw \]

\[ \rightarrow mq\zeta\alpha \int_{0}^{w} \int_{0}^{w} [S(w) - S(\tilde{q})] \, d\tilde{F}(\tilde{q}) w^{-\zeta-1} \, dw \]

Let \( \kappa \equiv \theta \int_{0}^{\infty} \int_{0}^{w} [S(w) - S(\tilde{q})] \, d\tilde{F}(\tilde{q}) w^{-\zeta-1} \, dw \) so that the surplus can be written as

\[ S(q) = \left(\frac{q}{Q}\right)^{\varepsilon-1} + \frac{mq\zeta\alpha}{\theta} \kappa \quad (19) \]

To derive an expression for \( \kappa \), we can use the expression for \( S(q) \) and plug this into the definition of \( \kappa \).

\[ \kappa = \theta \int_{0}^{w} \int_{0}^{w} S(w) F(\tilde{q}) \zeta w^{-\zeta-1} \, dw - \theta \int_{0}^{w} \int_{0}^{w} S(\tilde{q}) d\tilde{F}(\tilde{q}) w^{-\zeta-1} \, dw \]

\[ = \theta \int_{0}^{w} S(w) F(w) \zeta w^{-\zeta-1} \, dw - \theta \int_{0}^{w} \int_{\tilde{q}}^{\infty} \zeta w^{-\zeta-1} \, dw S(\tilde{q}) d\tilde{F}(\tilde{q}) \]

\[ = \theta \int_{0}^{w} S(w) F(w) \zeta w^{-\zeta-1} \, dw - \theta \int_{0}^{w} \tilde{q}^{-\zeta} S(\tilde{q}) d\tilde{F}(\tilde{q}) \]

Plugging in the expression for \( S \) from equation (19) and integrating gives

\[ \kappa = \frac{\theta^{\varepsilon-1} \Gamma\left(1 - \frac{\varepsilon-1}{\zeta}\right)}{(\varepsilon-1)Q^{\varepsilon-1}} + \frac{m \Gamma(1 - \alpha)}{\theta^{1-\alpha}} \kappa + \frac{\theta^{\varepsilon-1} \Gamma\left(2 - \frac{\varepsilon-1}{\zeta}\right)}{(\varepsilon-1)Q^{\varepsilon-1}} + \frac{m \Gamma(2 - \alpha)}{\theta^{1-\alpha}} \kappa = \frac{1}{\zeta} + \alpha \kappa \]

Thus \( \kappa = \frac{1}{1-\alpha} \frac{1}{\zeta}. \)