Consumer Referrals

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Abstract

In many industries, firms reward their customers for making referrals. We analyze the optimal policy mix of price, advertising intensity, and a referral fee for monopoly when buyers choose to what extent to refer other consumers to the firm. We find that the firm advertises less under referrals, but does not change its price from the monopoly level in an attempt to manage consumer referrals. We show that referral programs are Pareto-improving and that the firm underprovides referrals while supporting the socially optimum level of advertising. We extend the analysis to the case where consumer referrals can be targeted.

Keywords: consumer referral policy, word of mouth, referral reward program, targeted advertising, product awareness.

JEL numbers: C7, D4, D8, L1.

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1 Introduction

Firms often pay existing customers for referring potential customers to the firms’ products or services. Such referral programs are advertised as “Win/Win/Win” because existing customers, potential customers, and the firm can all benefit from referrals. The recent explosion in consumer-generated media, together with the documented trust consumers have in recommendations of other people, explains why firms would strive to manage word of mouth. Indeed, as consumer forums, blogs, and other means of consumer interaction are enhanced by advances in technology, the mix of information channels that firms rely on is changing. Firms are shifting away from mass advertising to more targeted channels that rely on consumers to spread the word about products. Improved online referral systems promise to make it easier for firms to monitor and control referral activity. How can firms use these new capabilities in designing their promotional strategies?

The adoption of a referral policy is one way firms can try to harness the power of word of mouth. A consumer referral policy is a promise by a firm to reward its customers for referring other people to the firm. For example, DIRECTV’ Referral Offer promises a $100 credit to any customers for referring a friend who signs up for the company’s service. Referral policies are adopted in a variety of industries, including banking, health care, Web design services, home remodeling, housing, vacation packages, home alarm systems, and high-speed Internet connection. They are used in the recruitment of nurses, technicians, and US Army

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1 According to Nielsen’s 2012 Global Trust in Advertising Survey of 28,000 Internet respondents from 56 countries, consumers around the world continue to find recommendations from personal acquaintances by far the most credible: 92 percent of respondents trust ("completely" or "somewhat") recommendations from people they know and 90 percent find these recommendations ("highly" or "somewhat") relevant. In comparison, ads are found trustworthy or relevant by 30-50 percent of respondents, depending on the media.
personnel, as well as in selling cars, houses, and tickets to sports events. Private schools, doctors, and daycare centers give out referral bonuses as well.\footnote{In our analysis, we will focus on the common case of fixed referral fees paid out to existing customers for referring a new customer who buys the product. A casual observation of referral policies suggests that referral payments are usually made in the form of cash, deposit, gift certificate, bonus points, free product or service, or entry into a lottery.}

In this paper, we explore how referral policies can be set to maximize their potential in generating value for the firm. The problem of choosing an optimal referral policy is complicated by the fact that a firm’s pricing, advertising, and referral policies interact. In particular, a referral policy cannot be chosen in isolation from other marketing tools because it affects the effectiveness of these other tools and vice versa. The goal of this paper is to find the optimal marketing mix while recognizing the interaction between pricing, advertising, and referral strategies. Additionally, we allow consumers to decide on the intensity of their referral activity by weighing referral incentives and the costs of making referrals. To our knowledge, no other study has taken such a comprehensive approach to developing an analytical model of consumer referrals.

We introduce consumer referrals into a monopoly market with advertising. Consumers can become informed about the firm’s product either directly through advertisements or indirectly through consumer referrals. The referral policy provides a monetary reward (referral fee) to consumers who had purchased the product and made a successful referral that led to the referred consumer’s purchase of the product. Consumer valuations for the product are i.i.d. random variables drawn from a common distribution, and they are the private information of the consumers. Consumers can make multiple referrals at a common cost, receiving a referral fee for each successful referral. In the benchmark model, we assume that a consumer sends referrals randomly to all other consumers. As a result, in equilibrium there
is congestion in referral messages. The firm can manage referral incentives in our model by changing its marketing mix (price, advertising intensity, and referral fee).

In this framework, we answer such questions as: When would a firm support active consumer referrals? Would it set a higher or lower price under referrals? Would it engage in more or less advertising under referrals? What are the overall welfare effects of referral policies? We show that the firm chooses to rely on an active referral policy as long as the cost to consumers of making referrals is not too high. We find that the firm advertises less under referrals and uses referral fees to manage referral activity, keeping the price at the monopoly level. Importantly, when the firm chooses to support active consumer referrals, it benefits all consumers as well (in the ex ante sense), confirming the potential “Win/Win/Win” feature of referral programs.

In this benchmark model, we first characterize the consumer referral equilibrium for each combination of the firm’s policy mix (Proposition 1). Then, we characterize the optimal policy mix. We find that the profit-maximizing price is the standard monopoly price as long as the referral fee is chosen optimally (Proposition 2). This somewhat surprising result follows from the fact that the referral fee is used to control the referral reach, and once the referral fee is chosen optimally it does not generate any distortion in the choice of price. Another result is that the firm’s advertising intensity is reduced when consumer referrals are introduced (Proposition 3). We also show that the firm’s advertisement level is the same as the social optimum level, while the profit-maximizing referral reach is lower than the socially optimal one (Proposition 6).

Although our benchmark model provides several interesting results, the model lacks an important aspect of consumer referrals. One of the major motivations for a firm to adopt a
consumer referral program is consumer information advantage: consumers may have better information than the firm about other consumers’ valuations. If consumer referrals can be targeted, then it could be even more beneficial for the firm to use consumers to disseminate information about the product. We show that the main results of the paper extend to the case when consumers know whether other consumers are informed and/or are willing to purchase the product.

We also extend our model by allowing two groups of consumers: High-type and Low-type consumers. High-type consumers tend to have higher valuations than Low-type consumers in the sense of first-order stochastic dominance. Firms cannot distinguish between consumer types, but consumers can tell who belongs to their group. In such a case, the monopoly should have stronger incentives to use consumer referrals, especially if the High-type group’s valuations tend to be much higher than the Low-type’s. It turns out that having two types can complicate the analysis significantly, but if willingness-to-pay distributions are significantly different across types, then we can obtain a clear result. We find that if only High-type consumers get referrals in the case of targeted referrals, then the monopoly price is higher, the ratio of referral fee to product price is lower, and the advertising level is lower than in case of random referrals (Proposition 8). Intuitively, if consumers have better information, the monopoly relies less on advertising and more on referrals.

A few streams of literature are relevant to our model. First, there is the literature strictly on consumer referrals: Jun and Kim (2008), Byalogorsky et al. (2005), and Galeotti and Goyal (2009). Using consumer networks as a channel for information transmission, these papers analyze how the monopoly’s choice of price and referral incentives affects profits. Jun and Kim (2008) assume a finite chain of consumers with i.i.d. random valuations. Consumers
are rational and forward-looking—they consider the expected benefit from giving a referral when making their purchase decisions. The authors show that even though the firm sets a common price and referral fee, it will effectively price-discriminate between consumers located early in the chain (who are more valuable to the monopoly) and those later in the chain due to differences in the consumers’ purchase probabilities. Byalogorsky et al. (2005) take the same setup as Jun and Kim (2008) but adopt a behavioral assumption that consumers make referrals whenever they are sufficiently delighted, that is, whenever the expected utility from making a purchase and making a referral exceeds a critical level of "consumer delight." They explore whether the firm would choose to promote referrals by paying referral fees and/or reducing its price. Since consumer delight includes the net value of purchase, the firm would lower its price to promote more referrals. When consumers are easy to delight, a referral program would not be used because referrals would be made anyway. But, the firm would use both a positive referral fee and a lower price when consumers are not so easy to delight.

In these papers, consumers are assumed to be able to contact only the next-in-line consumer, ruling out referral congestion.

Galeotti and Goyal (2009) consider a more complex network model in which consumers make multiple referrals with no cost, allowing for referral congestion. In their basic model, the monopoly is assumed to send advertisements randomly, and only the receivers of ads can make consumer referrals. They analyze the optimal advertisement policy in this model and

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3 Arbatskaya and Konishi (2013) justify the tie-breaking rule used in the paper, showing that effective price discrimination is indeed a common feature of the model in the second-best environment (with a common referral fee and price for all consumers) and the first-best environment (with a sufficient number of policy tools).

4 These papers do not consider advertising as an alternative communication channel. In contrast, Mayzlin (2006) looks at the case where advertising and word of mouth are both used to influence consumer choice between vertically differentiated products. While in her model, consumers cannot distinguish between promotional chat and consumer recommendations, in our model, advertising and referrals are two distinct channels.
show that using consumer referrals would unambiguously increase profits, but an increase in the level of social interaction (described with a network structure) can increase or decrease the advertisement level and profits. They use a reduced form model to concentrate on the relationship between network structure and the optimal strategy of the firm and its profits. In contrast, we assume a simple network structure (a complete network) while we analyze the optimal combination of the firm’s strategies: product price, referral fee, and level of advertising.

The second stream of literature focuses on advertising and congestion. In his pioneering paper, Butters (1977) formulated a competitive model of advertising in which firms send a number of advertisements to consumers randomly, informing them about the existence of the product and its price. Butters shows that price dispersion occurs in equilibrium, and consumers who receive multiple ads choose the lowest-price firm. In Butters’s model, some portion of ads are wasted due to congestion, but the level of congestion (the number of ads) is socially optimal. Van Zandt (2004), Anderson and de Palma (2009), and Johnson (2013) present alternative information congestion models in which consumers ignore some advertisements they receive. They all show that as the number of ads decreases, both firms and consumers are better off due to a reduction in congestion, though the mechanisms by which this occurs differ. In contrast, in our model, referrals are subject to congestion because we endogenize the referral intensity. Despite the presence of referral congestion, we find that referrals are underprovided by the monopoly. This result is due to the non-appropriability of consumer surplus. In the literature with one information channel, the firm would similarly

5Van Zandt (2004) assumes that all consumers can process up to a certain number of ads, while Anderson and de Palma (2009) assume that a consumer’s cost of processing ads depends on the number of ads she receives. Johnson (2013) allows consumers to decide what fraction of ads to block.
underprovide informative advertising, e.g. as in Esteban et al. (2001).

Finally, there is the literature on targeted advertising. Van Zandt (2004) and Johnson (2013) assume that firms sell heterogeneous products and have some information about consumer preferences. They analyze targeted advertising policies in oligopolistic markets (Van Zandt, 2004) and competitive markets (Johnson, 2013). Although the mechanisms are different, both papers show that improved targeting increases firms’ profits and makes consumers better off. Esteban et al. (2001) consider a monopoly firm choosing between mass advertising and targeted advertising with various degrees of targeting. With targeted advertising, the number of wasted ads is reduced, while the monopoly power increases. The authors show that the latter welfare loss tends to exceed the former benefit. Galeotti and Moraga-Gonzalez (2008) analyze a simple oligopolistic model of targeted advertising and show that market segmentation generates higher profits in equilibrium.

All of these papers assume that firms possess information on consumer types and therefore can conduct targeted advertising. In Section 5 we study related models of targeted referrals. We assume that the firm has no ability to distinguish between consumer types while consumers have some information on other consumers’ types. The firm uses consumers’ superior information to indirectly target consumers by supporting referrals with referral fees. In equilibrium, consumers who are not in the targeted group for referrals are not likely to be informed about the product, and they are worse off when consumer referrals are used in addition to mass advertising.

In the next section, we take the consumer’s perspective and analyze the consumer decision to buy the firm’s product and refer other people. In Section 3, we look at the firm’s choice of price, advertising intensity, and referral fee. Section 4 presents welfare considerations, and
Section 5 deals with targeted referrals. We conclude in Section 6. The Appendix contains the more technical proofs.

2 The Basic Model of Consumer Referrals

In this section, we describe the equilibrium consumer referral behavior in a market served by a monopoly that sets a price, advertises its product, and pays referral bonuses to the existing customers for referring new customers.

There is a large number $N$ of consumers. Each consumer purchases at most one unit of the product. Consumers differ only in their willingness to pay, which is assumed to follow a cumulative distribution function $G$ on $[0, 1]$ with a continuous density function $g$. Let $\pi(p) = p(1 - G(p))$ denote the firm’s profit from selling to a consumer at price $p$. We assume that there exists a unique monopoly price $p^m = \arg \max_p (\pi(p))$ and let $\pi^m = p^m(1 - G(p^m))$ be the associated monopoly profit.

The firm’s policy is described by a triple $(p, a, r)$: its level of advertisement $a \in [0, 1]$ (a fraction of consumers reached by advertisements), its price $p \geq 0$, and a referral policy characterized by a referral fee $r \geq 0$. We assume that the firm cannot price-discriminate between consumers who receive an advertisement and those who are referred to the firm by other consumers.

Advertisements inform consumers about the firm’s policy. They are distributed to consumers uniformly at random. That is, the probability that a consumer becomes informed through advertising is independent of the consumer’s valuation of the product. The number of consumers informed through advertisements ("the informed") is $I = aN$ and the number of consumers who do not receive ads ("the uninformed") is $U = (1 - a)N$. Consumers who
become informed through ads buy the product at the stated price or remain inactive.\footnote{Since the monopoly does not price discriminate based on a consumer’s information source, waiting to buy by referral can never be better for a consumer than buying immediately. Therefore, we assume that if a consumer receives an advertisement and has a nonnegative benefit from the product, she purchases it rather than waits for a referral to purchase by referral. Also, in our model, the net expected benefit of referral in equilibrium is zero, and therefore it does not matter whether consumers know about the referral policy prior to purchase or not.} Without receiving an advertisement or referral, a consumer would not know about the product and so cannot purchase it.

After making a purchase, a consumer can attempt to collect referral fees by referring other people. The decision to make referrals is endogenous in the model. A consumer trades off the cost and the expected benefit of making referrals. Each referral attempt costs $c > 0$, which captures the cost of informing a contact about the product. On the benefit side, referral attempts can be successful or unsuccessful. We initially assume that a referrer does not know other people’s willingness-to-pay and whether they have been informed through advertisements. If a referrer’s contact has a low willingness-to-pay and/or is already informed, the referral attempt will not be successful. Furthermore, potential referrals may have been contacted by others and may choose a different referring person.

The number of informed consumers who purchase the product and are potential referrers is $n = (1 - G(p))aN$.\footnote{For any referrals to be given in the market, the number of referrers and well as the number of uninformed consumers have to be positive, that is, $n = (1 - G(p))aN > 0$ and $U = (1 - a)N > 0$. This implies that $a \in (0, 1)$ and $1 - G(p) > 0$ must hold for any referral activity to be supported in the market.} Referring consumers simultaneously and independently choose referral intensity, i.e. the fraction of consumers to refer. They send referrals at random but without contacting the same person more than once, and referrals sent by different referrers are independently distributed among all consumers. A consumer choosing an individual referral intensity $q \in [0, 1]$ contacts $qN$ distinct consumers, but some of these consumers may have received advertisements or may have received multiple referral attempts.
A consumer referral equilibrium is a strategy profile \( q^* = (q_1^*, ..., q_n^*) \in [0, 1]^n \) such that \( q_i^* \) is the best response to \( q_{-i}^* \) for all \( i = 1, ..., n \). The equilibrium (overall) referral intensity is described by the total number of referral messages sent by referrers as a fraction of \( N \), 

\[
S^* \equiv \sum_{i=1}^{n} q_i^*.
\]

In the symmetric consumer referral equilibrium, each of \( n \) referrers sends referrals to a fraction \( q^* \) of all the consumers, i.e. \( q_i^* = q^* \) for all \( i = 1, ..., n \), and \( S^* = nq^* \geq 0 \).

We will focus on the symmetric equilibrium. Although there also exist a continuum of asymmetric equilibria, the equilibrium referral intensity is identical in all equilibria.

The referral reach, \( R \), is a fraction of all consumers reached by referrals. When each of \( n \) consumers refers a fraction \( q \) of all consumers, it is described by \( R = 1 - (1 - q)^n \). We will assume that \( n \) is large and therefore we can use an approximation \( R = R(S) = 1 - e^{-S} \) in which referral reach is a function of the (overall) referral intensity \( S = qn \).

\[
R(S) = 1 - e^{-S} \tag{1}
\]

with \( R(S) \in [0, 1) \), \( R(0) = 0 \), \( R' > 0 \), \( R'' < 0 \), and \( \lim_{S \to \infty} R(S) = 1 \).

As referral intensity increases, an increasingly smaller fraction of referrals is successful.

The level of congestion in referral messages can be measured by a function

\[
\tilde{\varphi}(S) \equiv \frac{S}{R(S)} = \frac{S}{1 - e^{-S}}, \tag{2}
\]

which is a ratio of the number of referral messages sent by referrers to the number of referrals registered by consumers. The congestion function has intuitive properties: \( \tilde{\varphi}(S) \geq 1 \), \( \lim_{S \to 0} \tilde{\varphi}(S) = 1 \), \( \tilde{\varphi}'(S) > 0 \) and \( \tilde{\varphi}''(S) > 0 \).\(^8\)

\(^8\)Note that \( \tilde{\varphi}'(S) = \frac{1}{(1 - e^{-S})^2} \left( 1 - e^{-S} (1 + S) \right) > 0 \), where \( 1 - e^{-S} (1 + S) > 0 \) because it is zero at \( S = 0 \) and increasing for all \( S > 0 \). Also note that \( \tilde{\varphi}''(S) = \frac{e^{-S}}{(1 - e^{-S})^3} (S + 2e^{-S} + Se^{-S} - 2) > 0 \) because expression \( (S + 2e^{-S} + Se^{-S} - 2) \) equals zero at \( S = 0 \) and is increasing:

\[
\frac{\partial}{\partial S} \left( \frac{1}{(1 - e^{-S})^3} (S + 2e^{-S} + Se^{-S} - 2) \right) = 1 - (1 + S)e^{-S} > 0
\]
In what follows, it is more convenient for us to express referral intensity and the level of congestion as a function of referral reach \( R \):

\[
S(R) = -\ln (1 - R), \tag{3}
\]

and

\[
\varphi(R) = \tilde{\varphi}(S(R)) = S(R)/R = -\ln (1 - R)/R. \tag{4}
\]

It follows that \( S'(R) = \frac{1}{1-R} > 0 \) and \( S''(R) = \frac{1}{(1-R)^2} > 0 \), and congestion as a function of referral reach has the same properties as congestion as a function of referral intensity: \( \varphi(R) \geq 1 \), \( \lim_{R \to 0} \varphi(R) = 1 \), \( \varphi'(R) = \tilde{\varphi}'(S(R)) \ast S'(R) > 0 \), and \( \varphi''(R) = \left(\tilde{\varphi}''(S(R)) + \tilde{\varphi}'(S(R))\right)(1 - R)^{-2} > 0 \).

Proposition 1 demonstrates that there exists a unique symmetric consumer referral equilibrium and characterizes the equilibrium referral intensity.

**Proposition 1.** Suppose the firm chooses a marketing mix of price \( p \), advertising intensity \( a \), and referral fee \( r \). Then the equilibrium referral reach \( R^* = R^*(p, a, r) \) is implicitly defined by

\[
(1 - a)(1 - G(p))r = c\varphi(R^*) \tag{5}
\]

for all \( r \) above the critical level \( r_0 \equiv \frac{c}{(1-a)(1-G(p))} \), and no referrals are sustained for lower levels of referral fee. For \( r > r_0 \), the equilibrium referral intensity and reach are higher when referral fee \( r \) is higher and price \( p \), advertising intensity \( a \), and referral cost \( c \) are lower.

We find that as long as \( n \) is large, the referral reach is independent of \( n \). An interesting feature of the equilibrium is that each referring consumer is indifferent between sending and (which in turn is the case because \( 1 - (1 + S) e^{-S} \) is zero at \( S = 0 \) and \( \frac{\partial(1-(1+S) e^{-S})}{\partial S} = S e^{-S} > 0 \). Finally, \( \lim_{S \to 0} \tilde{\varphi}(S) = \lim_{S \to 0} \left( \frac{S}{1-e^{-S}} \right) = \lim_{S \to 0} \left( \frac{1}{e^{-S}} \right) = 1 \), where we used L’Hôpital’s Rule.)
not sending an additional referral. That is, each referrer is indifferent between \( q_i \in [0, 1] \) because in equilibrium the benefit from making referrals is equal to its cost. The benefit of giving a referral is the referral fee multiplied by the probability that the referral is received by an uninformed consumer willing to buy the product, \( b = (1 - a)(1 - G(p))r \). In equilibrium, the benefit equals the cost of reaching a consumer with a referral message, \( c \varphi(R^*) \), which depends on the level of referral congestion (the number of referral attempts required to reach an additional consumer with a referral). The same property holds at the aggregate level. The total cost from giving referrals is equal to the (expected) referral fees collected, \( N S^*c = N(1 - a)(1 - G(p))R^*r \), where \( S^* = S(R^*(p, a, r)) = -\ln(1 - R^*(p, a, r)) \) is the equilibrium referral intensity. In other words, the net benefit to consumers from making referrals is zero. The comparative statics results of Proposition 1 imply that the equilibrium referral intensity \( S^* \), reach \( R^* \), and congestion \( \varphi(R^*) = \frac{S^*}{R^*} \) are higher when referral cost, price, and advertising intensity are lower and the referral fee is higher. Intuitively, the factors that increase the benefit of making referrals or reduce its cost must increase congestion for consumers to remain indifferent between referring and not.\(^9\)

3 Monopoly Choice of Price, Advertising, and Referral Fee

In this section, we characterize the optimal (profit-maximizing) monopoly policy \((p^*, r^*, a^*)\) and conditions under which a firm would choose to support consumer referrals. We assume that the firm operates at zero unit cost of production and the standard monopoly profit

\(^9\)If referrals were made sequentially, then \( R^* \) defined in Proposition 1 would still describe the equilibrium referral reach, and the last consumer who tried to make a referral would be indifferent between making referral or not. However, in this case, some consumers may obtain a positive benefit from making referrals. For example, if the firm introduces a new product to the market, in the initial stage, the net referral benefit may be positive, but as congestion in the referral messages grows, the net benefit shrinks to zero.
is higher than referral cost, $\pi^u > c$. The cost of advertising per consumer is described by function $C(a)$, which is increasing at an increasing rate in the fraction of consumers reached, $a$, $C''(a) > 0$ and $C'''(a) > 0$. To guarantee the interior solution in the presence of referrals, we would additionally assume that $C''(0) < c$ and $C(a)$ is sufficiently convex: $C''(a) > \frac{c}{(1-a)^2}$.

Let us first derive the firm’s profits. The number of uninformed consumers who receive referrals is $(1-a)R^*N$, where $R^* = R^*(p, a, r)$ is the referral reach, which is positive for high enough referral fees: $r > r_0 \equiv \frac{c}{(1-a)(1-G(p))}$ and $R^* = 0$ otherwise. Then, the (per consumer) demand that the firm faces is

$$D(p) = \left(\frac{I}{N} + \frac{U}{N} R^*(p, a, r)\right) (1 - G(p)) = (a + (1-a)R^*(p, a, r)) (1 - G(p)),$$

where $I = aN$ is the number of the informed consumers and $U = (1-a)N$ is the number of uninformed consumers. Under an active referral policy, the firm’s (per consumer) profit is then

$$\Pi(p, a, r) = p(1 - G(p))a + (p - r)(1 - G(p))(1 - a)R^*(p, a, r) - C(a),$$

and it is $\Pi_0(p, a) = a\pi(p) - C(a)$ when there are no referrals, where $\pi(p) = p(1 - G(p))$.

We can rewrite the firm’s profit by using a property of the consumer referral equilibrium that $r(1 - G(p))(1 - a)R^* = cS^*.$

**Lemma 1.** A firm’s profit in the consumer referral equilibrium under the firm’s policy vector $(p, a, r)$ is

$$\Pi(p, a, r) = (a + (1-a)R^*(p, a, r)) \pi(p) - C(a) - cS(R^*(p, a, r)).$$

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The profits are equal to the revenue obtained from making a fraction \( A = a + (1 - a) R^* \) of consumers aware about the product through advertisements and referrals, net of the cost of advertising to fraction \( a \) of consumers and sending \( S(R^*) \) referrals. Importantly, the firm’s marginal cost of referrals equals the consumers’ cost of making referrals.

We will next describe the optimal marketing mix of the firm – its choice of price, advertising, and referral fee. When there are no referrals, the monopoly price is \( p^m \equiv \arg \max_p (\pi(p)) \). However, it is not clear if the optimal price \( p^* \) in the presence of consumer referrals is higher or lower than \( p^m \). On the one hand, referral fees raise the cost of selling the product for the firm, and we would expect the firm to have a higher price under referrals. On the other hand, from Proposition 1 a higher price means lower referral incentives, and therefore the firm may want to set a lower price to promote referrals. Perhaps surprisingly, we can give a definite answer to this question when the firm optimally chooses both \( p^* \) and \( r^* \). The two effects cancel each other when the firm is optimally choosing its price and referral fee, and consumer behavior is described by the referral equilibrium. Proposition 2 states that under referrals the firm continues to set the standard monopoly price \( p^m \), irrespective of the level of advertising.

**Proposition 2.** The firm sets the standard monopoly price \( p^m \) when it supports consumer referrals with an optimally chosen referral fee.

To see why this proposition holds, consider the following three first-order conditions that
characterize the profit-maximizing monopoly policy vector \((p, a, r)\):

\[
\frac{d\Pi}{dp} = (a + (1 - a) R^*) \pi' (p) + [(1 - a) \pi (p) - cS'(R^*)] \frac{\partial R^*}{\partial p} = 0
\]
\[
\frac{d\Pi}{da} = (1 - R^*) \pi (p) + [(1 - a) \pi (p) - cS'(R^*)] \frac{\partial R^*}{\partial a} - C'(a) = 0
\]
\[
\frac{d\Pi}{dr} = [(1 - a) \pi (p) - cS'(R^*)] \frac{\partial R^*}{\partial r} = 0
\]

From the last condition, we find that, as long as referral reach is responsive to referral fee \((\frac{\partial R^*}{\partial r} > 0\), which occurs when \(r > r_0\)), the referral fee has to be set by the firm to equalize the marginal benefit of expanding referral reach and the marginal cost of doing so:

\[
(1 - a) \pi (p) = cS'(R^*).
\]

This means that for an optimal referral fee, the marginal net benefit of expanding referral reach is zero.

Figure 1 illustrates the firm’s choice of referral reach \(R^*\) that equalizes the marginal benefit of extending referral reach \(MB_R = (1 - a) \pi (p)\) and its marginal cost, \(MC_R = cS'(R) = c/(1 - R)\). Although the firm cannot directly control referral reach, it can set referral fee \(r\) in such a way as to achieve \(R^*\) through endogenous consumer referral decisions, which are based on the (average) cost and benefit of making a referral in equilibrium, \(AB_R = b = r(1 - a)(1 - G(p))\) and \(AC_R = c\phi(R)\).

We can use the observation that referral expansion has zero first-order effects on profits to further characterize the optimal choice of price and advertising. Using equation (12), the
first-order conditions for $p$ and $a$ can be written as

$$\frac{d\Pi}{dp}|_{r^*} = (a + (1-a) R^*) \pi'(p) = 0$$  \hspace{1cm} (13)$$

$$\frac{d\Pi}{da}|_{r^*} = (1 - R^*) \pi'(p) - C'(a) = 0$$  \hspace{1cm} (14)$$

Active consumer referrals imply that $a \in (0, 1)$ and $R^* > 0$. This means that the profit-maximizing price satisfies $\pi'(p) = 0$, and the firm’s pricing policy under active consumer referrals remains unchanged from the standard monopoly pricing, $p^* = p^m$. As is easily understood from the above derivations, although $p$ would affect the referral reach $R^*$, the monopoly can always control $R^*$ by adjusting referral fee $r$ optimally. That is, if the firm has the right policy tools, it can follow the standard monopoly pricing despite the fact that the price level affects the equilibrium referral reach.

Next we turn to firm’s advertising strategy. For an optimally set referral fee, the firm advertise less under referrals. This result follows from the comparison of first-order conditions under no referrals and under the optimal referral policy. Since we assume $C''(a) > 0$, $C''(a) > 0$, $C''(0) < c$, and $\pi^m > c$, there exists a unique unique interior profit-maximizing level of advertising under no referrals for $p = p^m$: $\bar{a} = \bar{a}(p) \equiv \arg \max \{a \pi(p) - C(a)\} \in (0, 1)$. It is described by the first-order condition $\frac{\partial \Pi}{\partial a} = \pi(p) - C'(a) = 0$. Compared to $\frac{d\Pi}{da}|_{r^*} = (1 - R^*) \pi(p) - C'(a) = 0$, the marginal cost of advertising is the same, while the marginal benefit of advertising, $(1 - R^*) \pi(p)$, is lower when the seller supports active consumer referrals because ads are wasted on consumers who become aware of the product through referrals. Hence, the firm advertises less with an optimally set referral fee than when it does not support referrals, and the result does not depend on a price level.

The next proposition shows that the firm advertises less even if the referral fee is set at
an arbitrary level \( r_0 < r < p \).

**Proposition 3.** The firm advertises less when it supports consumer referrals than when it does not, and the result is true for any price and referral fee consistent with active referrals.

There are two reasons why the firm would cut on advertising expenditures. First, less advertising means more uninformed consumers who can potentially become informed through referrals. Second, more uninformed consumers implies higher referral incentives and a higher proportion of consumers receiving referrals.

Lemma 2 describes the comparative statics responses of the optimal monopoly policy \((p^*, a^*, r^*)\) to changes in referral cost. From Proposition 2 we know that the optimal price does not change, but what happens to the referral fee and advertising?

**Lemma 2.** Under the optimal monopoly policy \((p^*, a^*, r^*)\) that supports consumer referrals, a higher referral cost results in a higher referral fee, more advertising, and lower referral reach. The comparative statics for referral fee and reach continue to hold when advertising intensity and/or price are exogenously fixed at arbitrary levels consistent with consumer referrals.

Intuitively, the firm offers a higher referral fee to compensate consumers for a higher referral cost. On the other hand, when it is not costly for consumers to refer their contacts, more of them attempt to make referrals, which results in a higher level of congestion in referrals. The firm responds to this by lowering incentives for referrals – it reduces the referral fee and increases its advertising level, leaving fewer consumers uninformed.

Since \( S'(R^*) = \frac{1}{1-R^*} \), we can describe the firm’s optimal choice of advertising intensity and referral fee (referral reach) as follows:
\[(1 - R^*) \pi (p) = C'(a) \]  
\[(1 - a)(1 - R^*) \pi (p) - c = 0 \]  

Note that the choices of \(a\) and \(R\) are interrelated. A higher level of advertising reduces the benefit from referrals and vice versa. This suggests that the two information channels are substitutes. Indeed, as Lemma 2 shows, when the referral cost increases, the firm supports fewer referrals and increases its reliance on advertising.

Proposition 4 provides a sufficient condition for the firm to use active consumer referrals.

**Proposition 4.** The optimal monopoly policy \((p^*, a^*, r^*)\) supports consumer referrals if the referral cost is sufficiently small,

\[ c < (1 - \bar{a}(p^m))\pi^m, \]  

where \(\bar{a}(p) = C'^{-1}(\pi(p))\).

The result is intuitive. In the proof, we note that the condition on the referral cost can be written as \(p^m > \frac{c}{(1 - \pi(p^m))(1 - G(p^m))}\). From Proposition 2, \(\bar{a}(p^m) > a^*\), and therefore \(p^m > \frac{c}{(1 - a^*)(1 - G(p^m))}\). When this inequality holds, the firm can set a referral fee slightly below the monopoly price to support active consumer referrals and earn more profits from consumers to the firm through referral.

We can also show that for any given \(a \in (0, 1)\) and \(p > 0\), the firm supports consumer referrals if and only if \(c < (1 - a)\pi (p)\). This follows from the fact that the inequality is equivalent to \(p > \frac{c}{(1 - a)(1 - G(p))}\), and therefore by Proposition 1, there exists referral fee \(r\) such that \(p > r > r_0 \equiv \frac{c}{(1 - a)(1 - G(p))}\). This referral fee supports consumer referrals and allows the
firm to gain additional profits from consumers coming by referral. The opposite is also true. 

If \( p \leq \frac{c}{(1-a)(1-G(p))} \), then there does not exist \( r \) such that \( p > r > r_0 \), and therefore the firm cannot increase its profits by the introduction of a referral program, keeping \( a \) and \( p \) fixed.

4 Welfare Considerations

Proposition 5 shows that allowing a firm to support consumer referrals generally results in a Pareto improvement.

**Proposition 5.** The equilibrium allocation achieved under the optimal monopoly policy \( (p^*, a^*, r^*) \) that supports consumer referrals Pareto-dominates the one achieved when the firm cannot use consumer referrals.

**Proof.** The firm cannot be worse off if it chooses to support referrals. Consumer demand expands and the monopoly price is unchanged under referrals. Hence, both the firm and consumers benefit from the presence of consumer referrals.

Given that the price is unchanged and more consumers are aware of the product under referrals, there is no consumer who is worse off. It follows that if the firm supports consumer referrals, it is socially optimal to do so.

To obtain further welfare results, we need to define a few terms. Consumer awareness of the product, \( A(p, a, r) = a + (1-a) R^*(p, a, r) \), is the measure of consumers informed through advertising or consumer referrals. From this definition, \( R^* = \frac{A-a}{1-a} \), and \( (1-A) = (1-a) (1-R^*) \) is the measure of consumers who remain unaware of the product. In what follows, we consider an equilibrium with active consumer referrals, and therefore we assume that the firm’s policy \( (p, a, r) \) is such that \( p > r > r_0 \equiv \frac{c}{(1-a)(1-G(p))} \) holds.
Monopoly profits are

$$\Pi(p, a, r) = A(p, a, r) \cdot \pi(p) - C(a) - cS(R^*(p, a, r)).$$  \hfill (18)$$

Let $CS(p) = \int_p^1 (v - p) g(v) dv$ be the (gross) consumer surplus from buying the product. Since the net benefit to consumers from making referrals is zero, consumer welfare is $CW = A(p, a, r) \cdot (CS(p) - \pi(p))$. Consumers benefit from higher product awareness and also prefer a lower price. We define the social welfare as the sum of monopoly profits and consumer welfare, $W = \Pi + CW$. The social welfare is then

$$W = A(p, a, r) \cdot CS(p) - C(a) - cS(R^*(p, a, r)).$$  \hfill (19)$$

Monopoly profits can be written in terms of price, advertising, and awareness:

$$\tilde{\Pi}(p, a, A) = A \cdot \pi(p) - C(a) - cS \left( \frac{A - a}{1 - a} \right).$$  \hfill (20)$$

How does the firm choose the awareness level? The first-order condition for awareness is

$$\frac{d\tilde{\Pi}}{dA} = \pi(p) - cS' \left( \frac{1}{1 - a} \right) = \pi(p) - \frac{c}{1 - A} = 0;$$  \hfill (21)$$

where we used $S' = \frac{1}{1 - R} = \frac{1 - a}{1 - A}$. The first-order condition just says that the marginal benefit of raising awareness is equal to the effective marginal cost (which is the cost of reaching a marginal consumer unaware of the product with a referral). That is, the profit-maximizing level of awareness is

$$A^* = 1 - \frac{c}{\pi(p)}.$$  \hfill (22)$$

For an optimally chosen referral fee, the consumer awareness $A^*$ does not depend on the level of advertising when it is exogenous to the model. It depends only on the profitability of sales $\pi(p)$ and cost of referral $c$. By similar arguments, the socially optimal level of awareness is

$$A^{SO} = 1 - \frac{c}{CS(p)},$$  \hfill (23)$$
and we conclude that the firm underprovides product awareness, $A^* < A^{SO}$.

[Figure 2 HERE]

Figure 2 compares the choices of product awareness by the firm and society. In both cases, the marginal cost is the cost of reaching an unaware consumer with a referral, $\frac{c}{1-A}$. Since the marginal social benefit of awareness $MSB_A = CS(p)$ exceeds the marginal benefit of raising awareness for the firm, $MB_A = \pi(p)$, product awareness is underprovided in the market.

How does the firm choose the advertising level? The first-order condition for advertising level $a$ is

$$\frac{d\Pi}{da} = -C'(a) + cS' \frac{1 - A}{(1 - a)^2} = -C'(a) + \frac{c}{(1 - a)} = 0.$$  \hspace{1cm} (24)

The firm’s choice of advertising under active referrals is guided by a simple cost analysis. The firm chooses the level of advertising to minimize the cost of advertising and referral payments, $C(a) + cS(R^*(a, r, p))c$, which is why it chooses the socially optimal level of advertising. This is true even if price is not set at the profit-maximizing level.

Figure 3 illustrates the trade-offs for the firm when selecting the level of advertising under an optimally chosen active referral policy. The firm sets the marginal cost of advertising equal to the cost of reaching an uniformed consumer with a referral. The advertising level is chosen to set the marginal cost of advertising $MC_a = C'(a)$ equal to the marginal benefit of advertising, which under active referrals is the cost of reaching an uniformed consumer with a referral, $MB_a = \frac{c}{(1-a)}$. There exists a unique interior solution because we assume $C'(0) < c$ and $C(a)$ is sufficiently convex: $C''(a) > \frac{c}{(1-a)^2}$. Conditional on the optimally
chosen referral fee, society would face the exact same trade-offs and choose the same level of advertising, \( a^{SO} = a^* \).

[Figure 3 HERE]

Given that the monopoly underprovides product awareness and faces the social trade-offs in selection of its advertising level, it follows that the firm supports fewer consumer referrals than is socially optimal, \( R^* < R^{SO} \). Proposition 6 summarizes the comparisons between socially optimal and profit-maximizing levels of advertising, referral reach, and product awareness.

**Proposition 6.** The monopoly policy that supports consumer referrals provides the socially optimal level of advertising but lower referral reach and awareness than is socially optimal.

Since the gains to society from higher product awareness are higher than the benefit to the firm, \( CS(p) > \pi(p) \), the firm underprovides product awareness. This is the standard result of non-appropriability of consumer surplus. The firm chooses the socially optimal advertising level because the choice of advertising is motivated by arguments of efficiency. The firm minimizes the cost of making consumers aware of the product using two information channels – advertising and consumer referrals. Therefore, the trade-off that the firm faces is the same as that under social-welfare maximization. As usual, the monopoly charges a price \( p = p^m \) that is higher than the socially optimal price, and the regular social inefficiency due to monopoly power occurs.
5 Better Informed Consumers

In the benchmark model, we assumed that consumers are no better informed than the monopoly about other consumers. Clearly, this may not be the case. Consumers may have superior knowledge regarding the values of other consumers and their product awareness. This gives the firm an additional incentive to rely on consumer referrals. In this section, we consider two simple extensions in this direction.

5.1 Single-type consumers

First, we consider how the results of the benchmark model change if (case 1) consumers know whether other consumers receive the firm’s advertisements and (case 2) consumers know other consumers’ valuations of for the product (or know that the valuations are high enough for consumers to buy the product). The first scenario may arise when consumers are known to subscribe to the same media or be members of a forum, newsgroup, or distribution list. The second scenario could be due consumers’ being informed about others’ income or preferences. It is also possible for consumers to know both 1) and 2) (case 3).

In general, the results of the benchmark model extend in these cases. In each case, there exists a unique referral equilibrium in which consumers receive zero net benefit from making referrals due to congestion in referral messages. Since referrals are not wasted on unlikely prospects, the firm has an incentive to rely more on targeted referrals. This makes advertising less attractive because ads are wasted on consumers who obtain referrals. Therefore, the firm has an incentive to reduce its level of advertising. The only difference between profits and social welfare is that the firm derives benefit \( \pi(p) \) from a sale while the benefit is \( CS(p) \) for the society as a whole. As in the benchmark case, the firm underprovides referral reach and
product awareness.

The results differ from the benchmark model for two reasons. First, when consumers are better informed, they target referrals to individuals who are more likely to buy the product. This implies that the benefit from each referral is higher. As a result, consumers choose to offer more referrals, and the level of congestion is higher in the referral equilibrium with better-informed consumers. This means that although referral messages are not wasted on unlikely prospects, because of the higher congestion level the savings are not fully captured by either consumers or the firm. To reduce congestion, the monopoly sets the referral-fee-to-price ratio at a lower level when referrals are targeted than when they are random. Second, a decision variable may affect the targetability of referrals. In cases 1 and 3, since the targetability of referrals can be improved by additional advertising, the comparison between the advertising level under random and targeted referrals is harder to make. The firm would prefer less advertising because targeted referrals are cheaper, and it can save money by doing less advertising and more referrals. At the same time, advertising is beneficial because it increases the targetability of referrals, making referrals even cheaper to use.

Proposition 7 summarizes our findings for targeted referrals for single-type consumers.

**Proposition 7.**

1). The monopoly supports more referrals and higher product awareness while offering a lower referral fee to price ratio when referrals are targeted than when they are random, as in the benchmark model.

2). When consumers know whether other consumers are informed through advertising, the firm’s price is the same as in the benchmark model. It advertises less than in the case of no
referrals.

3). When consumers know other consumers’ valuations, the firm’s price is higher and there is less advertising than in the benchmark model.

4). When consumers know who is informed and who is willing to buy the product, the firm’s price is higher than in the benchmark model. The firm advertises less than in the case of no referrals.

Consider first the case when consumers know whether other consumers are informed through advertising. In this case, the targetability of referrals is affected by advertising but not by price. As a result, the firm continues to charge the standard monopoly price. It advertise less than in the case of no referrals. The higher level of congestion in the referral equilibrium is combatted by a lower referral fee.

When consumers know other consumers’ valuations for the product, they refer only those contacts whose valuations exceed the price of the product. The firm’s price is higher than the standard monopoly price because pricing does affect the targetability of referrals: a higher price implies fewer consumers with sufficiently high valuations. Thus, there is an additional benefit to a price increase: more precise targeting of referrals and a resulting cost savings because referrals are not wasted on consumers who are not going to buy the product by referral. Less advertising is sustained in this case than in the benchmark model because referrals are more targeted and therefore cheaper to use when raising product awareness. An additional reason for less advertising is that because of the price distortion, the profitability of a sale is lower.

In case 3, referrals are given only to people who are uninformed and willing to buy the
product. In this case, both advertising and pricing affect the targetability of referrals, and therefore the monopoly has an incentive to distort them relative to the benchmark model in the direction of increasing targetability of referrals. That is, price is set above the monopoly level. Although monopoly has an incentive to increase advertising to make consumer referrals more targeted, it continues to advertise less than in the case of no referrals.

5.2 Two types of consumers

We continue to assume that the monopoly knows the cumulative distribution function of consumer willingness-to-pay for the general population, \( G \). The departure from the basic model is to assume that there are two groups of consumers: \( H \) and \( L \) with fractions \( \alpha^H \) and \( \alpha^L \), respectively \( (\alpha^H + \alpha^L = 1) \). We assume that consumers can tell which group other consumers belong to, while the firm cannot distinguish these two groups of consumers. Although consumers are better informed, the exact willingness-to-pay of each consumer is still that consumer’s private information.

Group \( H \) (\( L \)) consumers tend to have higher (lower) willingness-to-pay in the sense of the first-order stochastic dominance: i.e., for all \( v \), \( G^H(v) < G^L(v) \), where \( G^H(v) \) and \( G^L(v) \) are the cumulative distribution functions for groups \( H \) and \( L \). The supports of the distribution functions overlap, so that some consumers who belong to group \( L \) have higher willingness-to-pay than some consumers in group \( H \). The general distribution \( G(v) \) is a weighted average of \( G^H(v) \) and \( G^L(v) \): \( G(v) = \alpha^H G^H(v) + \alpha^L G^L(v) \) for all \( v \). We assume strict concavity of profit for each group: \(-2g^\theta(p) - p \times g^\theta(p) < 0\) for all \( p \) and all \( \theta \in \{H, L\} \).

Consumer \( i \) who received the firm’s advertisement can choose \( q_i^H \) and \( q_i^L \) as referral intensities for two different groups because she can distinguish which of her friends belong
to $H$ and $L$ groups. We have the same result as Proposition 1, but the condition applies for each group $\theta = H, L$. The equilibrium referral reach for group $\theta$ consumers $R^{\theta*} = R^{\theta*}(p, a, r)$ is defined implicitly by

$$(1 - a)(1 - G^\theta(p))r = c\varphi(R^{\theta*})$$

(25)

for all $r > r_0^\theta \equiv \frac{c}{(1-a)(1-G^\theta(p))}$, and the equilibrium referral intensity is higher when referral fee $r$ is higher and price $p$, advertising intensity $a$, and referral cost $c$ are lower. Given first-order stochastic dominance, we have $r_0^H < r_0^L$ for all $p, a$, and $r$ and $R^L* < R^H*$ for all $p, a$, and $r > r_0^H$. Therefore, referral congestion is higher in group $H$ than in group $L$, and it is possible that consumer referrals reach only group $H$ (if $r_0^H < r \leq r_0^L$ holds, which occurs when $1-G^H(p)$ is significantly larger than $1-G^L(p)$). The total cost of giving referrals to consumers in group $\theta$ is equal to the (expected) referral fees, $\alpha^\theta NS^{\theta*}c = \alpha^\theta N(1-a)(1-G^\theta(p))R^{\theta*}r$

Note that $\alpha^\theta$ has no effect in determining consumer referral intensity and reach in each group.

The (per consumer) monopoly profit in this environment is

$$\Pi(p, a, r) = p \sum_{\theta \in \{L, H\}} \alpha^\theta (a + (1-a)R^{\theta*}) (1 - G^\theta(p))$$

(26)

$$-c \sum_{\theta \in \{L, H\}} \alpha^\theta S^{\theta*} - C(a),$$

where $R^{\theta*} = R^{\theta*}(p, a, r) > 0$ for $r > r_0^\theta$ and $R^{\theta*} = 0$ otherwise.

Equation (26) is clearly a natural extension of Lemma 1, but there is an important difference. Although $S^{\theta*} = \ln(1 - R^{\theta*})$ is still completely determined only by $R^{\theta*}$, the firm can no longer control $R^{H*}$ and $R^{L*}$ independently by using a single referral fee $r$. Indeed, if we look at the first-order condition with respect to $r$, we have

$$\frac{d\Pi}{dr} = \sum_{\theta \in \{L, H\}} \alpha^\theta \left[ (1-a)x^\theta(p) - cS^\theta(R^{\theta*}) \right] \frac{\partial R^{\theta*}}{\partial r} = 0,$$

(27)
where \( \pi^\theta(p) = p(1 - G^\theta(p)) \). This formula is similar to equation (11), but \((1 - a)\pi^\theta(p) - cS^\theta(R^\theta) = 0\) is not assured for either \( \theta \). Thus, we cannot use the technique we used in the benchmark model to simplify \( \frac{dT}{dp} \) and \( \frac{dT}{da} \).

Of course, if the firm can use a differentiated referral fee system (\( r^H \) and \( r^L \)), the optimal referral reach \( R^H* \) and \( R^L* \) can be set for each group, and \((1 - a)\pi^\theta(p) - cS^\theta(R^\theta) = 0\). However, it is unreasonable to assume that the firm can set type-dependent referral fees because the whole point of this extension is to examine how the firm may use consumer referrals to utilize consumer information. For this reason, calculating the optimal monopoly price under active referrals for both groups is no longer simple. There is no dichotomy in the firm’s decision problem: use \( p \) to maximize profit per consumer and use \( r \) to control \( R^* \).

However, we can show that the firm would choose to increase its price after the introduction of active consumer referrals when only group \( H \) gets consumer referrals (i.e., when \( r^H_0 < r \leq r^L_0 \)). In this case, \( r \) needs to control only \( R^H* \), and we can apply the same technique as before.

We compare the optimal policies under random and targeted referrals, \((p^m, r^*, a^*)\) and \((p^T*, a^T*, r^T*)\), respectively. We will assume the following sufficient condition for no referrals to be extended to type-L consumers under targeted referrals: \( p^m(1 - G^L(p^m)) \leq c \).

**Proposition 8.** Suppose that \( p^m(1 - G^L(p^m)) \leq c \) holds. Then, under targeted referrals, the firm’s optimal policy \((p^T*, a^T*, r^T*)\) is such that group-L consumers receive no referrals, and the firm advertises less under targeted referrals than under random referrals, \( a^T* < a^* \). Moreover, the optimal monopoly price is higher under targeted referrals than the standard
monopoly price, \( p^{T*} > p^m \), if the hazard rates satisfy the following condition:

\[
\frac{g^H(p)}{1 - G^H(p)} < \frac{g(p)}{1 - G(p)}
\]

for all \( p \in (0, 1) \). The equilibrium referral reach is higher while the ratio of referral fee to product price is lower under targeted referrals than under random referrals, \( R^{T*} > R^* \) and \( r^{T*}/p^{T*} < r^*/p^m \).

The above hazard rate condition is a natural assumption. It means that the standard monopoly price for \( H \)-type consumers is higher than that for all consumers. Thus, in the case of targeted referrals, we no longer have the standard monopoly price. Notice, however, that as the proof of Proposition 8 shows, the optimal price is still the monopoly price given the set of consumers who get information through advertisements and referrals.

Proposition 8 shows that if consumers possess superior information about who would be likely to purchase the product, then the firm would reduce its reliance on mass advertising and shift to using more consumer referrals. Interestingly, consumers can be better off or worse off by the firm’s use of referrals when consumers have information advantage. Under no referrals, every consumer has an equal probability of receiving information about the product. However, with targeted referrals, consumers who belong to a low willingness-to-pay type are less likely to receive the information, although some of them may have high valuations of the product. Thus, the impact of targeted referrals on consumers may depend on consumer type.
6 Conclusion

Several information channels are available to sellers who market their products to consumers. These include traditional mass advertising on TV and in newspapers, targeted promotional advertising, "buzz" marketing, consumer word of mouth, and consumer referral policies. We look at the optimal marketing mix between advertising, referral policy, and price promotions, and discuss the welfare impact of referrals.

We find conditions for monopoly to support active referrals and characterize the equilibrium. Not surprisingly, the firm’s advertising intensity is reduced when consumer referrals are introduced. Perhaps surprisingly, we find that the profit-maximizing price is the standard monopoly price as long as the referral fee is chosen optimally. Intuitively, monopoly does not change its price in an attempt to manage consumer referrals but instead uses a referral fee. We also show that the firm’s advertisement level is the same as the social optimum level, while the profit-maximizing referral reach and product awareness are lower than the socially optimal levels.

Although our benchmark model provides multiple interesting results, the model lacks an important aspect of consumer referrals. One of the major motivations for a firm to adopt a consumer referral program is the consumer information advantage: consumers may have better information than the firm about other consumers’ valuations. We show that if consumer referrals can be targeted, then it could be even more beneficial for the firm to use consumers to disseminate information about the product.
References


Appendix: Proofs

Proposition 1. Proof. Consider the informed consumers’ equilibrium choice of referral intensity $q$. There are $n = (1 - G(p))aN > 0$ referrers (informed consumers who purchase the product) and $U = (1 - a)N > 0$ uninformed consumers, where $n$ is large. Referral attempts are made randomly. With probability $1 - a$ referral attempts reach the uninformed consumers. We assume that if a consumer receives $k$ referral attempts, then she chooses one with equal probability $1/k$. Focusing on a symmetric equilibrium, suppose that $n - 1$ referrers are choosing referral intensity $q$, while the remaining referrer $i$ chooses $q_i$. Then, the proportion of the uninformed consumers who use referrals from $i$ is

$$ F_i(q_i, q) = \sum_{k=0}^{n-1} \frac{1}{k+1} q_i (1 - q)^{n-1-k} q^k \times C(n-1, k), \quad (28) $$

where $C(n-1, k) = (n-1)!/(n-1-k)!k!$. Note that the term $(1 - q)^{n-1-k} q^k \times C(n-1, k)$ denotes the probability that an uninformed consumer receives $k$ referral attempts from other $n - 1$ referrers. By rearranging the formula, we obtain

$$ F_i(q_i, q) = \sum_{k=1}^{n} \frac{1}{k} q_i (1 - q)^{n-k} q^{k-1} \times C(n-1, k-1) \quad \text{(29)} $$

for $q > 0$, and $F_i(q_i, 0) = q_i$ for $q = 0$. Referrer $i$’s optimal referral choice $q_i$ is obtained by
solving

$$\max_{q_i} \left( (1 - G(p)) \times r \times m \times \frac{q_i}{nq} [1 - (1 - q)^n] - cNq_i \right)$$

(30)

for \( q \in (0, 1] \) and

$$\max_{q_i} ((1 - G(p)) \times r \times U \times q_i - cNq_i)$$

(31)

for \( q = 0 \), where the term \( 1 - G(p) \) denotes the probability that an uninformed consumer purchases the product and \( cNq_i \) denotes the cost of referring a fraction \( q_i \) of all consumers.

Since \( U = (1 - a)N \), we obtain

$$\max_{q_i} q_i N \left( (1 - G(p)) (1 - a) r \frac{[1 - (1 - q)^n]}{nq} - c \right)$$

(32)

for \( q \in (0, 1] \) and

$$\max_{q_i} q_i N ((1 - G(p)) (1 - a) r - c)$$

(33)

for \( q = 0 \).

Referrer \( i \)'s objective function is linear in \( q_i \). Note that \( [1 - (1 - q)^n] / nq < 1 \) for \( q \in (0, 1] \) and \( n \geq 2 \).\(^{10}\) For \( r \leq \frac{c}{(1 - a)(1 - G(p))} \) and \( q \in (0, 1] \), the unique best response is \( q_i = 0 \). Thus, in this case, \( q^* \in (0, 1] \) cannot be the equilibrium referral intensity in a symmetric equilibrium, and the only symmetric equilibrium is \( q^* = 0 \).

In a symmetric interior equilibrium, consumers are indifferent among all \( q_i \)s. The symmetric equilibrium \( q^* \) is implicitly calculated as a unique solution to

$$\frac{nq^*}{1 - e^{-nq^*}} = \frac{(1 - a)(1 - G(p)) r}{c}.$$
where \( c > 0, \ a > 0, \ p \geq 0, \) and \( r \geq 0. \) We used the approximation \( 1 - (1 - q)^n \approx 1 - e^{-nq} \) for small \( q^* \) because \( \log(1 - q)^n = n \log(1 - q) \approx -nq, \) and \( (1 - q)^n \approx e^{-nq} \) for small \( q. \)

The only candidate for the symmetric equilibrium intensity is \( S^* = nq^* \) that solves 
\[
\tilde{\varphi}(S^*) \equiv \frac{S^*}{1 - e^{-S^*}} = \frac{(1-a)(1-G(p))}{e} r.
\]
Since \( \tilde{\varphi}(S^*) \geq 1, \) this equation has a solution only if 
\[
r > \frac{c}{(1-a)(1-G(p))}.
\]
Then, indeed, given that others are choosing \( q^* \), consumer \( i \) obtains a zero payoff for any strategy, and she might as well choose \( q^* \). Thus, \( q^* \) is the symmetric referral equilibrium and \( S^* > 0 \) is the equilibrium referral intensity when \( r > \frac{c}{(1-a)(1-G(p))}. \) The equilibrium referral intensity \( S^* \) is unique when it exists because \( \tilde{\varphi}(S^*) \) is a strictly increasing function, \( \tilde{\varphi}'(S^*) > 0. \) In equilibrium it equals \( \frac{(1-a)(1-G(p))}{e} r, \) which is increasing in \( r \) and decreasing in \( a, \ p, \) and \( c. \) Hence, the equilibrium referral intensity, \( S^* = S^*(p, a, r) = nq^*, \)
is increasing in \( r \) and decreasing in \( a, \ p, \) and \( c: \frac{\partial S^*}{\partial r} > 0, \frac{\partial S^*}{\partial a} < 0, \frac{\partial S^*}{\partial p} < 0, \) and \( \frac{\partial S^*}{\partial c} < 0 \) for 
\[
r > \frac{c}{(1-a)(1-G(p))}.
\]
The equilibrium referral reach is \( R^* = R(S^*) = 1 - e^{-S^*}. \) Since it is an increasing function of \( S^* \), the comparative statics results for \( R^* \) are the same as those for \( S^*\).

**Proposition 3.** **Proof.** Without referrals, the monopoly profit is \( \Pi_0(p, a) = \pi(p) a - C(a), \)
and the marginal profit with respect to the level of advertising \( a \) is \( \frac{\partial \Pi_0}{\partial a} = \pi(p) - C'(a). \)
Assuming \( C'(0) < \pi(p), \ C'(a) > 0 \) and \( C''(a) > 0, \) there exists a unique profit-maximizing level of advertising under no referrals: \( \pi = C''^{-1}(\pi(p)). \)

With referrals, the monopoly profit is

\[
\Pi(p, a, r) = \Pi_0(p, a) + (p - r)(1 - G(p))(1 - a)R^*(p, a, r) \tag{35}
\]
and the marginal profit from advertising is

\[
\frac{d\Pi}{da} = \frac{\partial \Pi_0}{\partial a} + (p - r)(1 - G(p)) \left(-R^* + (1 - a) \frac{\partial R^*}{\partial a}\right). \tag{36}
\]
For \( p > r \), the second term is negative since \( R^* > 0 \) and by Proposition 1 \( \frac{\partial R^*}{\partial a} < 0 \) under consumer referrals. Hence, the monopoly would advertise less when it supports referrals, \( a^* < \bar{a} \), and the result holds for any \( p \) and \( r \) consistent with active referrals (a sufficient condition for that is \( p > r > \frac{c}{(1-a)(1-G(p))} \)).

The second-order sufficient condition for \( a \) given any \( p \) and \( r \) is

\[
\frac{d^2 \Pi}{da^2} = -C''(a) + (p-r)(1-G(p)) \left( -2\frac{\partial R^*}{\partial a} + (1-a) \frac{\partial^2 R^*}{\partial a^2} \right) < 0 \tag{37}
\]

It is satisfied for a sufficiently convex cost of advertising \( C'(a) \), but it is harder to satisfy than in the case of no referrals since the second term is positive. We can see that by totally differentiating \((1-a)(1-G(p))r = c\varphi(R^*)\) with respect to \( a \), we obtain \(-(1-G(p))r = c\varphi'\frac{\partial R^*}{\partial a}\), and therefore \( \text{sign}(\frac{\partial R^*}{\partial a}) = -\text{sign}(\varphi') < 0 \). Differentiating once more yields \( 0 = c\varphi''\frac{\partial R^*}{\partial a} + c\varphi'\frac{\partial^2 R^*}{\partial a^2} \). Hence, \( \frac{\partial^2 R^*}{\partial a^2} = -\frac{\partial R^*}{\partial a}\frac{\varphi''}{\varphi'} > 0 \).

For an optimally chosen referral fee, the second-order sufficient condition for \( a \) is satisfied if \( C''(a) > \frac{c}{(1-a)^2} \), and an additional assumption \( 0 < C'(0) < c \) guarantees the existence of a unique interior \( a^* \).

**Lemma 2. Proof.** From the first-order condition for referral fee \( r^* \), the equilibrium referral reach is

\[
R^*(p, r^*(p, a; c), a) = 1 - \frac{c}{(1-a)\pi(p)} \tag{38}
\]

and \( R^* > 0 \) because \( p > r^* > r_0 \equiv \frac{c}{(1-a)(1-G(p))} \) when the firm supports active consumer referrals. Since \( S^* = -\ln(1 - R^*) \), it follows that \( S^* = \ln \left( \frac{(1-a)c}{c} \pi(p) \right) \).

From Proposition 1, in the referral equilibrium,

\[
r^* = \frac{S^*}{R^* \frac{c}{(1-a)(1-G(p))}} \tag{39}
\]
Rewriting this expression using $R^*$ and $S^*$ for an optimally chosen $r^*$, we obtain

$$r^* = p\frac{\ln\left(\frac{(1-a)p}{c}\right)}{\frac{(1-a)p}{c} - 1}. \quad (40)$$

To see that $r^* = r^*(p, a; c)$ is an increasing function of $c$ and $a$, let $x \equiv \frac{\pi(p)(1-a)}{c}$. Then, $r^* = p\frac{\ln(x)}{x-1}$, and $\ln(x) = (1 - \frac{1}{x} - \ln x)(1-x)^{-2} < 0$ for $x > 0$. Hence, when advertising intensity is fixed, $r^*$ is increasing in $c$.

Next, assume that the firm could adjust advertising in response to changes in referral fees. The optimal $a$ and $r$ are such that $\pi(p) (1 - R^*) - C'(a) = 0$ and $\pi(p) (1 - R^*) (1 - a^*) = c$. Hence, $(1 - a^*) C'(a^*) = c$. Totally differentiating this identity with respect to $c$, we obtain $1 = (C''(a^*) (1 - a^*) - C'(a^*)) \frac{\partial a^*}{\partial c}$. It follows that $\frac{\partial a^*}{\partial c} > 0$ because $C(a)$ is assumed to be sufficiently convex to satisfy the second-order condition: $C''(a^*) \frac{C'(a^*)}{1-a^*} = \frac{c}{(1-a^*)^2}$.

We can now evaluate the total effect of a change in referral cost on referral fees. Since $r^* = p\frac{\ln x}{x-1}$ is decreasing in $x = \frac{\pi(p)(1-a^*)}{c}$, the price is independent of $c$, and $\frac{1-a^*}{c} = C'(a^*)$, we find that $\text{sign} \left( \frac{dr^*}{dc} \right) = \text{sign} \left( C''(a^*) \frac{\partial a^*}{\partial c} \right)$. To conclude, we find that the optimal referral fee is increasing in the referral cost, $\frac{dr^*}{dc} > 0$. The result does not depend on a price level, as long as $p > r^*$, and in particular holds for an optimally chosen price level $p^m$.

**Proposition 4. Proof.** By Proposition 2, the firm chooses price $p^m$ regardless of whether referrals are present. From Proposition 3 it follows that $a^* < \pi(p^m)$, and therefore the fraction of the uninformed is higher under referrals $1 - a^* > 1 - \bar{a}$. Suppose referral cost is sufficiently low: $c < (1-\bar{a}(p^m))p^m(1-G(p^m))$. Then, there exists $r \in \left( \frac{c}{(1-\pi(p^m))(1-G(p^m))} p^m \right)$, and from Proposition 1, active referrals are supported for such $r$: $R^* > 0$. The firm receives
positive additional profits from consumers coming by referral, without altering its profits from the informed consumers. Thus, when referral cost is sufficiently low, the monopoly profits improve by introducing a referral policy.■

**Proposition 7. Proof.** Let subscripts 1, 2, and 3 refer to cases 1) consumers know whether other consumers are informed through advertising; 2) consumers know other consumers’ valuations; and 3) the combination of 1) and 2): consumers know who is informed and who is willing to buy the product. Case 0 refers here to the benchmark model. We compare profit-maximizing policies under random referrals (case 0) \((p^*, a^*, r^*)\), and under targeted referrals (cases 1 through 3): \((p_1, a_1, r_1)\), \((p_2, a_2, r_2)\), and \((p_3, a_3, r_3)\).

In all cases, the benefit to consumers from making referrals is equal to the cost of referrals. This can be written as

\[
\begin{align*}
\text{case 0: } (1 - a)(1 - G(p))r R^* &= cS^* \\
\text{case 1: } (1 - a)(1 - G(p))r R^* &= (1 - a)cS^* \\
\text{case 2: } (1 - a)(1 - G(p))r R^* &= (1 - G(p))cS^* \\
\text{case 3: } (1 - a)(1 - G(p))r R^* &= (1 - a)(1 - G(p))cS^* 
\end{align*}
\]  

(41)

Active consumer referrals occur when the referral fee is above a case-dependent critical level \(r_0\):

\[
\begin{align*}
\text{case 0: } r_0 &= \frac{c}{(1-a)(1-G(p))} \\
\text{case 1: } r_0 &= \frac{c}{1-G(p)} \\
\text{case 2: } r_0 &= \frac{c}{1-a} \\
\text{case 3: } r_0 &= c 
\end{align*}
\]  

(42)

We can use equalities (41) to rewrite the firm’s profits as follows:

\[
\begin{align*}
\text{case 0: } \Pi(p, a, r) &= (a + (1 - a)R^*) \pi(p) - cS^* - C(a) \\
\text{case 1: } \Pi(p, a, r) &= (a + (1 - a)R^*) \pi(p) - (1 - a)cS^* - C(a) \\
\text{case 2: } \Pi(p, a, r) &= (a + (1 - a)R^*) \pi(p) - (1 - G(p))cS^* - C(a) \\
\text{case 3: } \Pi(p, a, r) &= (a + (1 - a)R^*) \pi(p) - (1 - a)(1 - G(p))cS^* - C(a) 
\end{align*}
\]  

(43)
In all cases, the optimal choice of \( r^* > r_0 \) is such that the marginal benefit to the firm of extending referral reach is equal to its marginal cost, and therefore:

\[
\begin{align*}
\text{case 0:} & \quad (1-a) (1-R^*) \pi(p) = c \\
\text{case 1:} & \quad (1-a) (1-R^*) \pi(p) = (1-a)c \\
\text{case 2:} & \quad (1-a) (1-R^*) \pi(p) = (1-G(p))c \\
\text{case 3:} & \quad (1-a) (1-R^*) \pi(p) = (1-a)(1-G(p))c
\end{align*}
\] (44)

When the referral fee is chosen optimally, the first-order conditions for price are

\[
\begin{align*}
\text{cases 0 and 1:} & \quad \pi'(p) A = 0 \\
\text{case 2:} & \quad \pi'(p) A + g(p) c S(R^*) = 0 \\
\text{case 3:} & \quad \pi'(p) A + g(p) (1-a)c S(R^*) = 0
\end{align*}
\] (45)

where \( A = a + (1-a)R^* \) is the overall product awareness.

Hence, the profit-maximizing price is \( p^* = p_1 = p^m \) in cases 0 and 1. Prices are higher in cases 2 and 3 than in the benchmark model: \( p_2 > p^m \) and \( p_3 > p^m \). It follows that \( \pi(p_2) < \pi^m \) and \( \pi(p_3) < \pi^m \), where \( \pi^m = \pi(p^m) \).

From (44), \( R^*_3 > R^*_1 > R^* \) because \( p_3 > p^m \) implies \( p_3 > \pi^m > (1-a^*)\pi^m \). As we will show next, \( a_2 < a^* \) and since \( p_2 > p^m > \pi^m \) we conclude that \( R^*_2 > R^* \). From (41) and (44),

\[
\frac{r}{p} = (1-R^*) \varphi(R^*)
\] (46)

in cases 0 through 3. Expression \( f(R) \equiv (1-R) \varphi(R) \) is a decreasing function of \( R \), where \( \varphi(R) = -\ln(1-R)/R \). To see this, differentiate \( f(R) \) to obtain \( f'(R) = \frac{1}{R^2} \ln(1-R) + \frac{1}{R} \).

Note that \( \ln x \) is a strictly concave function with \( \ln(1) = 0 \) and \( (\ln x)' = 1 \) at \( x = 1 \). Thus, \( \ln(x) < x - 1 \) for all \( x \neq 1 \). This implies \( \ln(1-R) < -R \), and for all \( R \in (0,1) \) we have \( f'(R) < -\frac{1}{R} + \frac{1}{R} = 0 \). Note also that \( f(1) = 0 \) and, using L’Hôpital’s Rule,

\[
\lim_{R \to 0} ((1-R) \varphi(R)) = \lim_{R \to 0} (1 + \ln (1-R)) = 1. \text{ Hence, } f(R) \in (0,1) \text{ for } R \in (0,1).
\]

From the properties of \( f(R) \) and (46) it follows that for any \( R^* \in (0,1) \), \( \frac{r}{p} \in (0,1) \), and \( r_1 < r^* \),

\[
\frac{r_2}{p^m} < \frac{r^*}{p^m}, \text{ and } \frac{r_3}{p^m} < \frac{r^*}{p^m}.
\]
When the referral fee is chosen optimally, the choices of advertising are governed by the first-order conditions:

\[
\begin{align*}
\text{case 0: } & \quad \frac{\partial}{\partial \alpha} = (1 - R^*) \pi (p) - C'(a) = 0 \\
\text{case 1: } & \quad \frac{\partial}{\partial \alpha} = (1 - R^*) \pi (p) + c S^* - C'(a) = 0 \\
\text{case 2: } & \quad \frac{\partial}{\partial \alpha} = (1 - R^*) \pi (p) - C'(a) = 0 \\
\text{case 3: } & \quad \frac{\partial}{\partial \alpha} = (1 - R^*) \pi (p) + (1 - G (p)) c S^* - C'(a) = 0
\end{align*}
\]

(47)

Using (44), we can write the first-order conditions for advertising as:

\[
\begin{align*}
\text{no referrals: } & \quad \frac{\partial}{\partial \alpha} = \pi (p) - C'(a) = 0 \\
\text{case 0: } & \quad \frac{\partial}{\partial \alpha} = \frac{c}{1 - a} - C'(a) = 0 \\
\text{case 1: } & \quad \frac{\partial}{\partial \alpha} = (1 + S^*) c - C'(a) = 0 \\
\text{case 2: } & \quad \frac{\partial}{\partial \alpha} = (1 - G (p)) \frac{c}{1 - a} - C'(a) = 0 \\
\text{case 3: } & \quad \frac{\partial}{\partial \alpha} = (1 - G (p)) (1 + S^*) c - C'(a) = 0
\end{align*}
\]

(48)

It immediately follows that \( a_2 < a^* \). While the cost of advertising \( C'(a) \) is common to all cases, the benefit of advertising at the profit-maximizing prices and referral reach differ:

\[
\begin{align*}
\text{no referrals: } & \quad \pi^m \\
\text{case 0: } & \quad \frac{c}{1 - a} \\
\text{case 1: } & \quad (1 + S^*_1) c \\
\text{case 2: } & \quad (1 - G (p_2)) \frac{c}{1 - a} \\
\text{case 3: } & \quad (1 - G (p_3)) (1 + S^*_3) c
\end{align*}
\]

(49)

Note that \( a_1 < \bar{a} (p^m) \) is equivalent to \( (1 + S^*_1) c < \pi^m \), or \( 1 - \ln(1 - R^*_1) < \frac{1}{(1 - a^*_1)(1 - R^*_1)} = 1/(1 - R^*_1) \). The inequality holds because \( f(R) = (1 - R) (1 - \ln(1 - R)) < 1 \) for \( R > 0 \)
due to \( f(0) = 1 \) and \( f'(R) = \ln(1 - R) < 0 \). Hence, \( a_1 < \bar{a} (p^m) \).

Note that \( a_3 < \bar{a} (p^m) \) is equivalent to \( (1 - G (p_3)) (1 + S^*_3) c < \pi^m \), or \( (1 - G (p_3)) (1 - \ln(1 - R^*_3)) = (1 - G (p_3)) (1 - \ln(c/p_3)) < \pi^m \). Since \( \pi^m > \pi (p_3) \), we only need to prove that \( 1 - \ln(c/p_3) < p_3 \). We can rewrite the last inequality as \( 1 - \ln(c) < p_3 - \ln(p_3) \). It suffices to show that \( x - \ln(x) > 1 \) for \( x > 0 \). Indeed, that is the case because \( (x - \ln(x))' = -\frac{1}{x} (x - 1) < 0 \) for \( x < 1 \) and \( (x - \ln(x))' = \frac{1}{x} (x - 1) > 0 \) for \( x > 1 \), and \( 1 - \ln(1) = 1 \). Thus, \( a_3 < \bar{a} (p^m) \).

From (44), the levels of awareness at the profit-maximizing solutions are described by:
We conclude that there is greater product awareness when referrals are targeted: $A_1 > A^*$, $A_2 > A^*$, and $A_3 > A^*$.

**Proposition 8. Proof.** By the assumption $p^m(1 - G^L (p^m)) \leq c$,

$$(1 - a)p(1 - G^L (p)) - \frac{c}{1 - R^L} \leq 0,$$

for all $p \geq p^m$, $a \in [0, 1)$, and $R^L \in [0, 1)$. That is, the marginal cost of extending referrals to type-$L$ consumers exceeds the marginal benefit.

Consider $p \geq p^m$. In the referral equilibrium under targeted consumer referrals,

$$(1 - a)(1 - G^H (p))rR^{T*} = cS^*(R^{T*})$$

if $r_0^H < r \leq r_0^L$. The firm’s profit function is

$$\Pi(p, a, r) = p(1 - G (p))a + \alpha^H p(1 - G^H (p))(1 - a)R^{T*} - \alpha^H cS^*(R^{T*}) - C(a).$$

The first-order condition for the referral fee is

$$\frac{d\Pi}{dr} = \alpha^H \left[ (1 - a)p(1 - G^H (p)) - \frac{c}{1 - R^{T*}} \right] \frac{\partial R^{T*}}{\partial r} = 0,$$

and therefore the expression in the square brackets is zero at the optimal $r^{T*}$ (that is, the marginal net benefit of extending referral reach among group-$H$ consumers is zero). If follows that

$$(1 - a) (1 - G^H (p))(1 - R^{T*})p = c.$$
First, consider pricing. Conditional on the optimal choice of $r^{T^*}$, the first-order condition for price is

$$\frac{d\Pi}{dp}_{r=r^{T^*}} = [1 - G(p) - pg(p)] a + \alpha^H [1 - G^H(p) - pg^H(p)] (1 - a) R^{T^*} = 0 \quad (55)$$

The standard monopoly price $p^m$ satisfies $1 - G(p^m) - p^m g(p^m) = 0$. Since $\frac{g^H(p)}{1-G^H(p)} < \frac{g(p)}{1-G(p)}$ holds for all $p$, we have $\frac{p^m g^H(p^m)}{1-G^H(p^m)} < \frac{p^m g(p^m)}{1-G(p^m)} = 1$. Thus, $1 - G^H(p^m) - p^m g^H(p^m) > 0$ holds, and we conclude that the firm indeed charges a price $p^{T^*}$ higher than $p^m$ when referrals become targeted. This argument justifies $p \geq p^m$.

Second, we consider advertising. Under the optimally chosen $r^{T^*}$, the derivative of the profit with respect to $a$ can be written as:

$$\left.\frac{d\Pi}{da}\right|_{r=r^{T^*}} = p (1 - G(p)) - \alpha^H p(1 - G^H(p)) R^{T^*} - C'(a) \quad (56)$$

$$= \alpha^L p(1 - G^L(p)) + \alpha^H \frac{c}{1-a} - C'(a). \quad (57)$$

In the benchmark model of random referrals, the profit-maximizing level of advertising $a^*$ (conditional on the optimal choice of $r^*$) is described by

$$p^m(1 - G(p^m)) (1 - R^*) - C'(a^*) = \frac{c}{1-a^*} - C'(a^*) = 0. \quad (58)$$

This means that the firm has lower incentives for advertising under targeted referrals, and $a^{T^*} < a^*$, since $\left.\frac{d\Pi}{da}\right|_{r=r^{T^*}} < 0$ holds under $p = p^{T^*}$ and $a = a^* > 0$ by:

$$\alpha^L p^{T^*}(1 - G^L(p^{T^*})) + \alpha^H \frac{c}{1-a^*} - C'(a^*) < \alpha^L p^m(1 - G^L(p^m)) + \alpha^H \frac{c}{1-a^*} - C'(a^*) \quad (59)$$

$$\leq \alpha^L c + \alpha^H \frac{c}{1-a^*} - C'(a^*)$$

$$< \frac{c}{1-a^*} - C'(a^*) = 0.$$
Third, we compare the optimal referral reaches under targeted and random consumer referrals ($R^T*$ and $R^*$, respectively):

$$(1 - a^*)p^m (1 - G(p^m))(1 - R^*) = c$$

(60)

and

$$(1 - a^T*)p^T*(1 - G^H(p^T*))(1 - R^T*) = c.$$  

(61)

These equations imply $R^T* > R^*$ because we know that $p^{T*} > p^m$, $a^{T*} < a^*$, and $p^{T*}(1 - G^H(p^{T*})) > p^m(1 - G^H(p^m)) > p^m(1 - G(p^m))$. To see the first inequality, note that

(i) $\frac{d(p(1-G^H(p)))}{dp} |_{p=p^{T*}} > 0$ holds from the above argument since $p^{T*} \in (p^m, p^{H*})$, where $p^{H*} = \arg \max_p \{p(1 - G^H(p))\}$, and (ii) profit function $p(1 - G^H(p))$ is assumed to be concave.

Finally, we will compare the ratios of referral fee and product price in the two cases. From Proposition 1, we obtain equilibrium referral condition for random consumer referrals:

$$(1 - a^*)(1 - G(p^m))r^* = c\phi (R^*) .$$

(62)

Similarly, under targeted referrals we obtain:

$$(1 - a^{T*})(1 - G^H(p^{T*}))r^{T*} = c\phi (R^{T*}) .$$

(63)

Using equations (60), (61), (62), and (63), we obtain:

$$\frac{r^*}{p^m} = (1 - R^*)\phi (R^*)$$  

(64)

$$\frac{r^{T*}}{p^{T*}} = (1 - R^{T*})\phi (R^{T*}) .$$

(65)

Since $R^{T*} > R^*$, to show that $\frac{r^*}{p^m} > \frac{r^{T*}}{p^{T*}}$, we only need to prove that $f(R) \equiv (1 - R)\phi (R)$ is a decreasing function. This is done after equation (46) in the proof of Proposition 7.
FIGURES

Figure 1. The Choice of Referral Reach

Notes: For a given level of advertising $a \in (0, 1)$, $MB_R = (1-a)\pi(p)$ and $MC_R = cS'(R) = c/(1-R)$ for the firm, and $AB_R = r(1-a)(1-G(p))$ and $AC_R = c\phi(R)$ for consumers.

Figure 2. The Social and Monopoly Choices of Product Awareness

Notes: $MB_A = \pi(p)$, $MSB_A = CS(p)$, and $MC_A = c/(1-A)$.
Figure 3. The Firm’s Marginal Cost and Benefit of Advertising

Notes: \( MB_a = \frac{c}{1-a} \) and \( MC_a = C'(a) \).