A Theory of Consumer Referral: Revisited

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Abstract

Jun and Kim (2008) consider the optimal pricing and referral strategy of a monopoly that uses a consumer communication network to spread product information. They show that for any finite referral chain, the optimal policy involves a referral fee that provides strictly positive referral incentives and effective price discrimination among consumers based on their positions in the chain. We revisit this problem to strengthen Jun and Kim’s results by weakening their referral condition. Moreover, we characterize the first-best policy when individual-specific referral fees are available and show that it is qualitatively similar to the second-best solution of Jun and Kim (2008).

Keywords: consumer referral policy, referral fee, price discrimination.

JEL numbers: D4, D8, L1.

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Highlights

- We strengthen the analysis of consumer referrals by Jun and Kim (IJIO 2008).
- In the model, a monopoly uses a referral chain to spread product information.
- We examine the optimal pricing and referral strategy of the firm.
- The firm indeed supports nonbinding referral conditions and price discrimination.
- The outcome in the second-best problem resembles that in the first-best problem.
1 Introduction

Jun and Kim (2008) consider the optimal pricing and referral strategy of a monopoly that uses a consumer referral chain to spread product information. In their model, one firm sells a product to a finite chain of \( n \) consumers. A consumer who purchases the product can refer it to the immediate successor. Each consumer’s willingness-to-pay \( v \) is an i.i.d. random variable drawn from a twice continuously differentiable distribution function \( F(v) \) over \([v, \bar{v}]\) with density function \( f(v) \). The monopoly firm can choose a price \( p \) and referral fee \( r \) that they pay for a successful referral: i.e., a consumer can receive a referral fee if the consumer refers the next-in-line consumer to the product, and that consumer purchases the product. Consumers need to pay a cost \( \rho > 0 \) to make a referral. Consumers make their purchase and referral decisions to maximize their expected utility.

In this model, the last consumer (consumer \( n \)) cannot make a referral, her purchase probability is \( \alpha_n = 1 - F(p) \), and her purchase and referral of the product do not generate extra sales. The second-to-last consumer’s purchase has an externality since her purchase may lead to consumer \( n \)’s purchase, but the externality is limited only to sales made to her successor. For consumers positioned earlier in the chain, the externality is larger. That is, early buyers are more valuable to the firm than later buyers since their purchase of the product is necessary for the referral chain to continue, and the potential gains from a longer chain are larger.

With this model, Jun and Kim first show that when the second-to-last consumer has a strictly positive referral benefit, \( r(1 - F(p)) > \rho \) (their referral condition RC), the earlier a consumer is located in the chain, the higher is her probability of purchasing the product \( \alpha_1 \) >
$\alpha_2 > \ldots > \alpha_n$ (their Proposition 1). This result further implies effective price discrimination among consumers according to their positions in the referral chain: although the firm charges a common price $p$ and pays a referral fee $r$ to all consumers, the firm effectively discriminates in favor of consumers located earlier in the chain because these consumers obtain a higher expected benefit from making a referral.\(^1\) Then, Jun and Kim numerically calculate the optimal price and referral fee combinations. Somewhat surprisingly, the optimal product price and referral fee are non-monotonic functions of the chain’s length, and referral and production costs $\rho$ and $c$.

We take a closer look at the optimal strategy of the firm. When the referral chain is endless ($n$ goes to infinity), it is easy to show that the firm’s optimal policy generates a stationary outcome (with equal purchase probabilities $\alpha_1 = \alpha_2 = \ldots = \alpha_n$).\(^2\) In the finite case, since $n$ variables $\alpha_1, \ldots, \alpha_n$ need to be controlled by two policy tools $p$ and $r$, finding an optimal choice of $(p, r)$ is a second-best problem, and it can be a highly nonconvex problem with multiple local maxima. This means that although intuitively it may be beneficial for the firm to (effectively) price discriminate between consumers based on their positions in a chain, the stationary outcome is another plausible candidate for the optimal solution, especially when $n$ is a large finite number.

A stationary outcome is characterized by a binding referral condition and no price discrimination. However, Jun and Kim (2008) say nothing about the case where the referral condition is binding: $r(1 - F(p)) = \rho$ (or $r\alpha_n = \rho$).\(^3\) It is easy to show that if the referral condition is binding: $r(1 - F(p)) = \rho$ (or $r\alpha_n = \rho$).

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\(^1\) Consumer $k$’s purchase probability $\alpha_k$ depends on consumer $(k + 1)$’s purchase probability $\alpha_{k+1}$ since consumer $k$ takes the expected net benefit from referral $r\alpha_{k+1} - \rho$ into account when she makes her purchase decision: $\alpha_k = 1 - F(p + r\alpha_{k+1} - \rho)$.

\(^2\) When there are infinite consumers in a chain, an indeterminacy problem arises, and there is a continuum of equilibrium strategies $(p, r)$ that support the unique optimal stationary $\alpha$. See Appendix C for details.

\(^3\) Jun and Kim (2008) assume that if a consumer is indifferent between making and not making a referral,
ral condition is binding even for one consumer, then purchase probabilities must be equal \( \alpha_1 = \alpha_2 = \ldots = \alpha_n \), which also implies that there is no price discrimination among consumers.

In this note, we examine the possibility of stationary outcome being optimal. We allow for referral equilibrium to be consistent with a binding referral condition \( r(1 - F(p)) = \rho \) by assuming that consumers make referrals when they are indifferent between making and not making referrals. We obtain two results that strengthen Jun and Kim’s findings. First, we show that the firm’s profit can be improved by increasing both \( p \) and \( r \) in a right proportion starting from the optimal stationary outcome, implying that the stationary outcome is not even a local maximum for any finite \( n \) (Theorem 1). This result strongly justifies Jun and Kim’s analysis, and also implies that at least for large \( n \), the optimal solution is perhaps very close to the stationary outcome. Second, we find that when the firm can charge different referral fees based on consumers’ positions (the first-best problem), both the probability of purchase \( \alpha_k \) and the expected referral fee \( r_k \alpha_k \) are decreasing as \( k \) increases (Theorem 2). This implies that Jun and Kim’s second-best solution is qualitatively close to the first-best solution, since both \( \alpha_k \) and \( r \alpha_k \) are decreasing in Jun and Kim’s solution (their Proposition 1). Despite the simplicity of the Jun and Kim model, the proofs are rather involved. Techniques developed in this note may be useful for other purposes.

2 Jun and Kim’s Problem

Here we show that the profit-maximizing stationary outcome is not a local maximum. Denote by \( \alpha_i \) the probability that consumer \( i \) buys the product conditional on being introduced to then she will not make a referral. This tie-breaking rule is convenient since it directly implies that consumers make referrals if and only if there are positive incentives for referral (their Proposition 2). Thus, a stationary outcome is ruled out as it is not compatible with active referrals. Here, we are assuming a tie-breaking rule that allows for referrals to be given in the stationary outcome. We assume that if a consumer is indifferent between making and not making a referral, she refers.
The firm chooses a strategy \((p, r)\) to maximize its profits

\[
\Pi(p, r) = (p - r\alpha_2 - c) \alpha_1 + (p - r\alpha_3 - c) \alpha_1\alpha_2 + ... \\
+ (p - r\alpha_n - c) \alpha_1 \cdots \alpha_{n-1} + (p - c)\alpha_1 \cdots \alpha_n
\]

(1)

where \(\alpha_1, ..., \alpha_n\) are determined by \((p, r)\) as follows: \(\alpha_n = D(p) = 1 - F(p) \geq 0\) and \(\alpha_k = D(p - \alpha_{k+1} r + \rho) \geq 0\) for \(k = 1, ..., n - 1\).

Denote by \(P(\alpha) = D^{-1}(\alpha)\) the standard inverse demand function. We assume that the profit function without referrals, \(\pi(\tilde{\alpha}) \equiv \tilde{\alpha} (P(\tilde{\alpha}) - c)\), is concave. Assuming \(r\alpha_k \geq \rho\) for \(k = 2, ..., n, \alpha_1, ..., \alpha_n\), are determined by the following system of equations:

\[
P(\alpha_n) = p \\
P(\alpha_{n-1}) = p - r\alpha_n + \rho \\
\vdots \\
P(\alpha_1) = p - r\alpha_2 + \rho,
\]

and \(\alpha_1 \geq ... \geq \alpha_n\). Suppose that the referral condition is binding for the \(k\)th consumer: \(r\alpha_{k+1} = \rho\) for some \(k = 1, ..., n - 1\). Then, \(P(\alpha_k) = p\), and we have \(P(\alpha_1) = P(\alpha_2) = \ldots = P(\alpha_n) = p\) and \(\alpha_1 = \alpha_2 = \ldots = \alpha_n\). This is a stationary outcome, for which consumer referral conditions are all binding: \(r\alpha_{k+1} = \rho\) for all \(k = 1, ..., n - 1\). We will show that this outcome is not locally optimal.

The firm’s profit can be written in terms of \(\alpha_k\)s only:

\[
\Pi(\alpha_1, ..., \alpha_{n-1}, \alpha_n; n) = \alpha_1 (P(\alpha_1) - \rho - c) + \alpha_1\alpha_2 (P(\alpha_2) - \rho - c) + ... \\
+ \alpha_1 \cdots \alpha_{n-1} (P(\alpha_{n-1}) - c - \rho) + \alpha_1 \cdots \alpha_n (P(\alpha_n) - c)
\]
where \((\alpha_1, \ldots, \alpha_n)\) is a solution to system (2). Under the stationary outcome \(\alpha_1 = \ldots = \alpha_{n-1} = \alpha_n = \alpha\), the monopoly profit when there are \(n \geq 1\) consumers can be written as

\[
\Pi(\alpha, \alpha, \ldots, \alpha; n) = A_n(\alpha) (\pi(\alpha) - \rho) + \rho, \tag{4}
\]

where

\[
A_n(\alpha) \equiv 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = \frac{1 - \alpha^n}{1 - \alpha} \tag{5}
\]

and \(\pi(\alpha) = \alpha (P(\alpha) - c)\).

Denote the optimal stationary policy for an \(n\)-consumer chain by \(\beta(n) \equiv \arg \max_\alpha \Pi(\alpha, \alpha, \ldots, \alpha; n)\). Theorem 1 states that \(\beta(n)\) cannot be a local maximum for small \(\rho\).

**Theorem 1.** The optimal stationary policy \(\beta(n)\) is not the optimal policy if \(\pi(\beta(n)) > \rho\).

For the formal proof, see Appendix A, where we show that the firm’s profit is locally improvable (starting from \(\beta(n)\)) by choosing an appropriate policy change \((dp, dr) \gg 0\). It follows from Theorem 1 that the optimal strategy \((p, r)\) is such that the referral condition is not binding for any consumer. This justifies the tie-breaking rule adopted by Jun and Kim (2008). As is known from Jun and Kim’s (2008) Proposition 1, this implies that the firm price-discriminates by subsidizing consumer referrals and generating \(\alpha_1 > \ldots > \alpha_n\).

We will provide a sketch of the proof of Theorem 1 here. First, in Lemma 1, we investigate the properties of the optimal stationary policy \(\beta(n)\). Then, we look at the profit function evaluated at the optimal stationary policy \(\alpha_1 = \ldots = \alpha_{n-1} = \alpha_n = \beta(n)\). We show that there is some \(M\) \((1 \leq M < n)\) such that profits increase with purchase probability for consumers located before \(M\) and decrease with purchase probability for consumers located after \(M\):

\[
\frac{\partial \Pi}{\partial \alpha_k} \Big|_{\alpha = \beta(n)} > 0 \text{ for all } k < M \quad \text{and} \quad \frac{\partial \Pi}{\partial \alpha_k} \Big|_{\alpha = \beta(n)} < 0 \text{ for all } k > M \tag{Lemma 2}.
\]

We then show...
that there exists a policy change $d\Delta = (dp, dr) \gg 0$ such that for any $M$ ($1 \leq M < n$) the probability of buying increases for consumers located before $M$ and decreases for consumers located after $M$. We prove this by showing that, starting at $\alpha_1 = \ldots = \alpha_n = \alpha$, if $\alpha_n$ decreases while $\alpha_M$ is kept constant, $d\alpha_k > 0$ for all $k < M$ and $d\alpha_k < 0$ for all $k > M$ (Lemma 3). Using Lemmas 2 and 3, we conclude that the optimal stationary policy is not a local maximum.

### 3 The First-Best Problem

In this section, we consider the same $n$-consumer model as Jun and Kim (2008) but allow for referral fees to vary along the referral chain. That is, the policy tools are $p$ and $r_2, \ldots, r_n$, where $r_k$ is the referral fee that consumer $k-1$ can get if consumer $k$ purchases the product following her referral.

Let $\alpha^m$ be the standard monopoly output, $\alpha^m = \arg\max \{\alpha (P(\alpha) - c)\}$, and $\pi^m = \alpha^m (P(\alpha^m) - c)$ be the associated monopoly profit. The monopoly profit with active consumer referrals is

$$
\Pi = \sum_{k=2}^{n} \left[ (p - c - r_k \alpha_k) \left( \prod_{\ell=1}^{k-1} \alpha_\ell \right) \right] + (p - c) \prod_{\ell=1}^{n} \alpha_\ell \tag{6}
$$

where $\alpha_k$ is consumer $k$’s probability of purchase when she is informed about the product: $\alpha_n = D(p)$ and $\alpha_k = D(p - \alpha_{k+1} r_{k+1} + \rho)$ for all $k = 1, \ldots, n - 1$.

We can describe the problem in terms of $\alpha_k$’s only by using

$$
p = P(\alpha_n) \tag{7}
$$

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4 After we completed the current paper, Jeong-Yoo Kim let us know about his unpublished note (Kim 2006) that analyzes the first-best problem for the special case of $n = 3$. 
and

\[ r_k = \frac{P(\alpha_n) - P(\alpha_{k-1}) + \rho}{\alpha_k} \quad (8) \]

for \( k = 2, \ldots, n \), where \( P(\alpha) = D^{-1}(\alpha) \). The firm’s profit can be written as

\[ \Pi = \sum_{k=2}^{n} \left[ (P(\alpha_{k-1}) - c - \rho) \left( \prod_{l=1}^{k-1} \alpha_l \right) \right] + (P(\alpha_n) - c) \prod_{l=1}^{n} \alpha_l \quad (9) \]

\[ = \sum_{k=1}^{n-1} \left[ (P(\alpha_k) - c - \rho) \left( \prod_{l=1}^{k} \alpha_l \right) \right] + (P(\alpha_n) - c) \prod_{l=1}^{n} \alpha_l \quad (10) \]

It is easy to see what the optimal probability of buying is for consumer \( n \). Taking the first-order condition with respect to \( \alpha_n \), we obtain

\[ \frac{\partial \Pi}{\partial \alpha} = (P(\alpha) - \alpha P'(\alpha) - c) \left( \prod_{l=1}^{n-1} \alpha_l \right) = 0. \quad (11) \]

This implies that \( \alpha_n = \alpha^m \) and \( P(\alpha_n) \) is the monopoly price for any \( n \) and \( k = 1, \ldots, n - 1 \) as long as \( \alpha_k > 0 \). This observation is quite sensible: no matter how many consumers are there, the last consumer does not make a referral, and the firm should charge the monopoly price for her. The main result we have is the following:

**Theorem 2.** Suppose \( \pi^m > \rho \). For all \( n \), the firm’s optimal policy satisfies \( \alpha_1^* > \alpha_2^* > \ldots > \alpha_n^* = \alpha^m \), \( p^* = P(\alpha^m) \), and the expected referral benefits for consumers 1 through \( n - 1 \) satisfy \( \alpha_2^* r_2^* > \alpha_3^* r_3^* > \ldots > \alpha_n^* r_n^* > 0 \).

The formal proof of Theorem 2 is in Appendix B, but we provide a sketch of the proof here. To prove Theorem 2, we analyze the situation where \( k \) consumers are left in the chain, and we solve recursively by backward induction. Let \( V(k) \) be the optimal profit from the last \( k \) consumers, and let \( \gamma^*(k) \) be the profit-maximizing purchase probability of the \( k \)th to last consumer. Since \( \gamma^*(1) = \alpha^m \), the expected profit from the last consumer reached by referral is the simple monopoly profit, \( V(1) = \pi^m \).
The optimal solutions $V(k)$ and $\gamma^*(k)$ when $k$ consumers are left in the chain can be defined recursively:

$$V(k) = \max_{\alpha} [\alpha (P(\alpha) - c - \rho) + \alpha V(k-1)]$$

and

$$\gamma^*(k) = \arg \max_{\alpha} [\alpha (P(\alpha) - c - \rho) + \alpha V(k-1)]$$

for all $k \geq 2$. Proposition 1 in Appendix B shows that the optimal purchase probability for the $k$th to last consumer $\gamma^*(k)$ is an increasing sequence of $k$: $\gamma^*(k + 1) > \gamma^*(k)$ for all $k$, assuming the optimal profit is increasing with the number of consumers left, i.e. $V(k + 1) > V(k)$ for all $k$.

Under the assumption of increasing profit sequence, we can use Proposition 1 to characterize the optimal purchase probability sequence: $\alpha_k^* = \gamma^*(n - k + 1)$ for any fixed $n \geq 2$ and all $k = 1, ..., n$ because the $k$th consumer from the top is the $(n - k + 1)$th consumer from the bottom: i.e., $\alpha_1^* > \alpha_2^* > ... > \alpha_n^*$ (Corollary 1 in Appendix B).

Proposition 2 in Appendix B shows that the optimal profit sequence $V(k)$ is indeed increasing as long as $\pi^m > \rho$. This proposition is proved by induction. Suppose that $V(k) > ... > V(1)$. Then, looking at the monopoly problem with $k + 1$ consumers, we show that the firm can achieve higher profits $V(k + 1) > V(k)$ if it applies the optimal policy for $k$ to the first $k$ consumers and provides consumer $k$ with just enough incentives to make a referral to consumer $k + 1$ (which is profitable because $\pi^m > \rho$ and consumer $k + 1$ will face the monopoly price). This proves that $V(k)$ is an increasing sequence. By putting Corollary 1 and Proposition 2 together, we conclude that $\alpha_1^* > \alpha_2^* > ... > \alpha_n^* = \alpha^m$, which also implies that $\alpha_2^* r_2^* > \alpha_3^* r_3^* > ... > \alpha_n^* r_n^* > 0$. 

9
4 Concluding Remarks

In the framework of Jun and Kim (2008), two qualitatively different referral equilibria could possibly arise. The one described by Jun and Kim (the non-stationary outcome) is characterized by a nonbinding referral condition, unequal probabilities of purchase, and price discrimination among consumers. The other one (i.e., the stationary outcome) involves a binding referral condition, equal probabilities of purchase, and no price discrimination among consumers. We strengthen Jun and Kim’s findings by showing that even if we allow for the stationary outcome to arise by adopting a natural tie-breaking rule for referrals, we can show that it cannot be optimal. We also show that the equilibrium in the second-best problem (with a common referral fee) resembles the equilibrium in the first-best problem (when the firm can set referral fees conditional on consumer location in the chain).

References


Appendix A

We prove Theorem 1 by using a sequence of lemmas.

**Lemma 1.**

(i) For all \( n \) and all \( \alpha \) such that \( \pi (\alpha) - \rho > 0 \), \( \Pi (\alpha, \alpha, ..., \alpha; n + 1) > \Pi (\alpha, \alpha, ..., \alpha; n) \).

(ii) The optimal stationary solution \( \beta (n) \equiv \arg \max_\alpha \Pi (\alpha, \alpha, ..., \alpha; n) \) satisfies the following condition:

\[
\pi' (\beta (n)) = -\frac{A'_n (\beta (n))}{A_n (\beta (n))} (\pi (\beta (n)) - \rho).
\]

(iii) Suppose \( \pi (\beta (n)) - \rho > 0 \). Then, \( \beta (n) > \beta (n - 1) > ... > \beta (1) = \arg \max_\alpha \pi (\alpha) \).

**Proof.** From (4), the difference in profits from \( (n + 1) \)- and \( n \)-consumer chains is

\[
\Delta^n (\alpha) \equiv \Pi (\alpha, \alpha, ..., \alpha; n + 1) - \Pi (\alpha, \alpha, ..., \alpha; n) = \alpha^n (\pi (\alpha) - \rho).
\]

Hence, \( \Delta^n (\alpha) > 0 \) if \( \pi (\alpha) - \rho > 0 \). This proves (i).

The optimal policy \( \alpha = \beta (n) \) is implicitly defined by the first-order condition

\[
\frac{d\Pi (\alpha, \alpha, ..., \alpha; n)}{d\alpha} = A'_n (\alpha) (\pi (\alpha) - \rho) + A_n (\alpha) \pi' (\alpha) = 0.
\]

This proves (ii).

Finally, using (14), we find that at \( \alpha = \beta (n) \)

\[
\frac{d\Delta^n}{d\alpha} = n\alpha^{n-1} (\pi (\alpha) - \rho) + \alpha^n \pi' (\alpha)
\]

\[
= \alpha^{n-1} (\pi (\alpha) - \rho) \left( \frac{nA_n (\alpha) - \alpha A'_n (\alpha)}{A_n (\alpha)} \right) > 0.
\]
The last inequality holds because

\[ nA_n(\alpha) - \alpha A'_{n}(\alpha) = n \left(1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1}\right) - \alpha \left(1 + 2\alpha + \ldots + (n-1)\alpha^{n-2}\right) \quad (18) \]

\[ = n + (n-1)\alpha + (n-2)\alpha^2 + \ldots + \alpha^{n-1} > 0. \]

Hence, if \( \alpha = \beta(n) > 0 \) and \( \pi(\beta(n)) - \rho > 0 \), then \( \beta(n+1) > \beta(n) \). Since \( \beta(1) = \alpha^m > 0 \), it follows that \( \pi'(\alpha) < 0 \) for all \( \alpha > \beta(1) \). Thus, \( \pi(\beta(1)) > \pi(\beta(2)) > \ldots > \pi(\beta(n)) \) holds, and we conclude that \( \beta(1) > \beta(2) > \ldots > \beta(n) \) if \( \pi(\beta(n)) - \rho > 0 \). \( \square \)

Notice that the profit \( \Pi(\alpha_1, \alpha_2, \ldots, \alpha_n; n) \) in equation (3) can be defined recursively:

\[ \Pi(\alpha_n; 1) = \pi(\alpha_n) = \alpha_n (P(\alpha_n) - c) \quad (19) \]

\[ \Pi(\alpha_{n-1}, \alpha_n; 2) = \alpha_{n-1} (P(\alpha_{n-1}) - c - \rho) + \alpha_{n-1} \Pi(\alpha_n; 1) \]

\[ \Pi(\alpha_{n-2}, \alpha_{n-1}, \alpha_n; 3) = \alpha_{n-2} (P(\alpha_{n-2}) - c - \rho) + \alpha_{n-2} \Pi(\alpha_{n-1}, \alpha_n; 2) \]

\[ \ldots \]

\[ \Pi(\alpha_1, \alpha_2, \ldots, \alpha_n; n) = \alpha_1 (P(\alpha_1) - c - \rho) + \alpha_1 \Pi(\alpha_2, \ldots, \alpha_n; n-1) \]

Using these formulas, we will prove the following result.

**Lemma 2.** Suppose that \( \pi(\beta(n)) - \rho > 0 \) holds. At \( \alpha_1 = \ldots = \alpha_n = \beta(n) \), (i) \( \frac{\partial \Pi}{\partial \alpha_n} < 0 \) and \( \frac{\partial \Pi}{\partial \alpha_1} > 0 \); (ii) there exists \( M \) such that \( \frac{\partial \Pi}{\partial \alpha_k} > 0 \) for any \( k < M \) and \( \frac{\partial \Pi}{\partial \alpha_k} < 0 \) for any \( k > M \).

**Proof.** The marginal profits with respect to buying probabilities \( \alpha_1, \ldots, \alpha_n \) are
\[
\frac{1}{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{n-1}} \frac{\partial \Pi}{\partial \alpha_n} = \pi'(\alpha_n)
\] (20)

\[
\frac{1}{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{n-2}} \frac{\partial \Pi}{\partial \alpha_{n-1}} = \pi'(\alpha_{n-1}) + \Pi(\alpha_{n-1}; 1) - \rho
\]

\[
\frac{1}{\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{n-3}} \frac{\partial \Pi}{\partial \alpha_{n-2}} = \pi'(\alpha_{n-2}) + \Pi(\alpha_{n-2}, \alpha_{n-1}; 2) - \rho
\]

\[
\ldots
\]

\[
\frac{\partial \Pi}{\partial \alpha_1} = \pi'(\alpha_1) + \Pi(\alpha_2, \ldots, \alpha_{n-1}; n-1) - \rho.
\]

Thus, at \(\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha\),

\[
\frac{\partial \Pi}{\partial \alpha_n} = \alpha^{n-1} \pi'(\alpha)
\] (21)

\[
\frac{\partial \Pi}{\partial \alpha_{n-1}} = \alpha^{n-2} (\pi'(\alpha) + \Pi(\alpha; 1) - \rho)
\]

\[
\ldots
\]

\[
\frac{\partial \Pi}{\partial \alpha_1} = \pi'(\alpha) + \Pi(\alpha, \ldots, \alpha; n-1) - \rho
\]

Note that \(\frac{\partial \Pi}{\partial \alpha_n} = \alpha^{n-1} \pi'(\alpha) < 0\) at \(\beta(n) > \beta(1) = \alpha^m\) by Lemma 1. By assumption, \(\Pi(\alpha; 1) = \pi(\beta(n)) > \rho\), and by Lemma 1, \(\Pi(\alpha, \ldots, \alpha; k)\) is increasing in \(k\). Hence, if for some \(\ell\), \(\frac{\partial \Pi}{\partial \alpha_\ell} > 0\), then \(\frac{\partial \Pi}{\partial \alpha_k} > 0\) for any \(k < \ell\), and if \(\frac{\partial \Pi}{\partial \alpha_\ell} < 0\), then \(\frac{\partial \Pi}{\partial \alpha_k} < 0\) for any \(k > \ell\).

In the following, we will show that for \(\alpha = \beta(n) > 0\), there exists \(M\) such that \(\frac{\partial \Pi}{\partial \alpha_k} > 0\) for any \(k < M\) and \(\frac{\partial \Pi}{\partial \alpha_k} < 0\) for any \(k > M\). Recall

\[
A_k(\alpha) = 1 + \alpha + \alpha^2 + \ldots + \alpha^{k-1} = \frac{1 - \alpha^k}{1 - \alpha}.
\] (22)

For \(k \leq n - 1\), we have

\[
\frac{\partial \Pi}{\partial \alpha_k} = \alpha^{k-2} (\pi'(\alpha) + \Pi(\alpha, \ldots, \alpha; n-k+1) - \rho)
\] (23)
where

$$\Pi(\alpha, \alpha, \ldots; n - k + 1) = A_{n-k+1}(\alpha) (\pi(\alpha) - \rho) + \rho. \quad (24)$$

Hence, $\frac{\partial \Pi}{\partial \alpha_k} > 0$ as long as

$$\pi'(\alpha) + A_{n-k+1}(\alpha) (\pi(\alpha) - \rho) > 0 \quad (25)$$

Plugging in the expression for the optimal $\pi'(\alpha)$ from Lemma 1 and assuming $\pi(\beta(n)) - \rho > 0$, the inequality is equivalent to

$$A_{n-k+1}(\alpha) A_n(\alpha) > A'_n(\alpha), \quad (26)$$

which is equivalent to

$$\frac{1 - \alpha^{n-k+1}}{1 - \alpha} \frac{1 - \alpha^n}{1 - \alpha} > \frac{\partial (\frac{1 - \alpha^n}{1 - \alpha})}{\partial \alpha} = \frac{1}{(1 - \alpha)^2} (1 - \alpha^n - n\alpha^{n-1} + n\alpha^n).$$

Thus, we conclude that $\frac{\partial \Pi}{\partial \alpha_k} > 0$ holds if and only if

$$\left[ n\alpha^{n-1} \frac{1 - \alpha}{1 - \alpha^n} - \alpha^{n-k+1} \right] > 0. \quad (27)$$

The contents of the brackets is strictly decreasing in $k$. Note that for $k = 1$, $[n\alpha^{n-1} \frac{1 - \alpha}{1 - \alpha^n} - \alpha^n] = (n \frac{1 - \alpha}{1 - \alpha^n} - \alpha) \alpha^{n-1} > 0$ because $n (1 - \alpha) - \alpha (1 - \alpha^n) > n (1 - \alpha) - \alpha (1 - \alpha) = (n - \alpha) (1 - \alpha) > 0$. Hence, $\frac{\partial \Pi}{\partial \alpha_1} > 0$. Since $\frac{\partial \Pi}{\partial \alpha_n} = \alpha^{n-1} \pi'(\alpha) < 0$, there exists $M \ (1 \leq M < n)$ such that $\frac{\partial \Pi}{\partial \alpha_k} > 0$ for any $k < M$ and $\frac{\partial \Pi}{\partial \alpha_k} < 0$ for any $k > M$ at $\alpha = \beta(n). \square$

In Lemma 3, we describe the effects of a policy change $(dp, dr) \gg 0$ at a stationary outcome $\alpha_1 = \ldots = \alpha_n = \alpha$.

**Lemma 3.** Consider a policy of increasing $p$ and $r$, starting at $\alpha_1 = \ldots = \alpha_n = \alpha$. For any $M \ (1 \leq M < n)$, there is a policy change $(dp, dr) \gg 0$ such that $d\alpha_k > 0$ for all $k < M$ and $d\alpha_k < 0$ for all $k > M$. 

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Proof. Totally differentiating equations (2) and evaluating at $\alpha_1 = ... = \alpha_n = \alpha$, we have:

$$P'(\alpha)d\alpha_n = dp$$  \hspace{1cm} (28)

$$P'(\alpha)d\alpha_{n-1} = (dp - \alpha dr) - r d\alpha_n$$

$$P'(\alpha)d\alpha_{n-2} = (dp - \alpha dr) - r d\alpha_{n-1}$$

$$\vdots$$

$$P'(\alpha)d\alpha_1 = (dp - \alpha dr) - r d\alpha_2.$$  

When $p$ is increasing ($dp > 0$), we necessarily have $d\alpha_n = \frac{1}{P'(\alpha)} dp < 0$.

Let $x \equiv \frac{p}{P'(\alpha)} > 0$. We choose $(dp, dr) \gg 0$ such that $d\alpha_M = 0$. From

$$P'(\alpha)d\alpha_M$$  \hspace{1cm} (29)

$$= (dp - \alpha dr) - r d\alpha_{M+1}$$

$$= (dp - \alpha dr) (1 + x) - r x d\alpha_{M+2}$$

$$= (dp - \alpha dr) (1 + x + ... + x^{n-M-1}) - r x^{n-M-1} d\alpha_n$$

$$= (dp - \alpha dr) (1 + x + ... + x^{n-M-1}) + x^{n-M} dp$$

$$= dp (1 + x + ... + x^{n-M}) - \alpha dr (1 + x + ... + x^{n-M-1}) = 0,$$

it follows that $d\alpha_M = 0$ implies

$$dp = \alpha dr \frac{1 + x + ... + x^{n-M-1}}{1 + x + ... + x^{n-M}}$$  \hspace{1cm} (30)

Similarly,

$$P'(\alpha)d\alpha_k = dp (1 + x + ... + x^{n-k}) - \alpha dr (1 + x + ... + x^{n-k-1})$$  \hspace{1cm} (31)
Using (30),

\[ P'(\alpha) d\alpha_k = \alpha dr \left[ \frac{1 + x + \ldots + x^{n-M-1}}{1 + x + \ldots + x^{n-M}} (1 + x + \ldots + x^{n-k}) - (1 + x + \ldots + x^{n-k-1}) \right]. \]

(32)

Then, since \( P'(\alpha) < 0 \) and \( dr > 0 \), \( d\alpha_k > 0 \) if and only if

\[ \frac{1 + x + \ldots + x^{n-M-1}}{1 + x + \ldots + x^{n-M}} < \frac{1 + x + \ldots + x^{n-k-1}}{1 + x + \ldots + x^{n-k}}. \]

(33)

This inequality holds whenever \( k < M \) because \( \frac{1 + x + \ldots + x^{n-1}}{1 + x + \ldots + x^n} = \frac{1 - x^n}{1 - x^{n+1}} \) and \( \frac{\partial}{\partial a} \left( \frac{1 - x^n}{1 - x^{n+1}} \right) = x^a \frac{\ln x}{(1-x^a+1)^2} (x - 1) > 0. \) Similarly, \( d\alpha_k < 0 \) whenever \( k > M. \)

From Lemmas 2 and 3, we conclude that, assuming \( \pi(\beta(n)) > \rho \), the optimal stationary outcome \( \beta(n) \) is not a local optimum for any finite \( n \). This proves Theorem 1.
Appendix B

Here we provide the proofs to Propositions 1 and 2, Corollary 1, and Theorem 2.

**Proposition 1.** Suppose that \( \pi^m > \rho \) and \( V(k + 1) > V(k) \) for any \( k \geq 1 \). Then, \( \gamma^*(k + 1) > \gamma^*(k) \) holds for all \( k \geq 1 \).

**Proof of Proposition 1.** We have four steps to show this:

1. Consider a chain of length \( n = 1 \). The profit-maximizing probability \( \gamma^*(1) = \alpha^n \) satisfies \( MR(\gamma^*(1)) = c \), where \( MR(\alpha) = (\alpha P(\alpha))' = P(\alpha) - \alpha P'(\alpha) \) is the marginal revenue from extending sales.

2. Next consider \( n = 2 \). In this case, the optimization problem is

\[
V(2) = \max_\alpha [\alpha (P(\alpha) - c - \rho) + \alpha V(1)],
\]

and \( V(2) > V(1) \) implies that there is an optimal solution \( \alpha^*(2) \) in this problem. Since \( MR(\alpha) \) is decreasing, the following first-order condition characterizes \( \gamma^*(2) \):

\[
MR(\gamma^*(2)) - c - \rho + V(1) = 0.
\]

Since \( V(1) = \pi^m > \rho \), \( MR(\alpha) \) is decreasing, and \( MR(\gamma^*(1)) - c = 0 \), we conclude that \( \gamma^*(2) > \gamma^*(1) \).

3. Suppose that \( \gamma^*(1) < ... < \gamma^*(k - 1) \) for \( k \geq 3 \). We will show that \( \gamma^*(k - 1) < \gamma^*(k) \) holds. By definition, we have

\[
V(k) = \max_\alpha [\alpha (P(\alpha) - c - \rho) + \alpha V(k - 1)],
\]
and the first-order condition for $\gamma^*(k)$ is

$$MR(\gamma^*(k)) - c + V(k - 1) - \rho = 0.$$ 

The first-order condition for $\gamma^*(k - 1)$ is

$$MR(\gamma^*(k - 1)) - c + V(k - 2) - \rho = 0.$$ 

Since $MR(\alpha)$ is decreasing and $V(k - 2) < V(k - 1)$, we conclude that $\gamma^*(k - 1) < \gamma^*(k)$ holds.

4. By an induction argument, we complete the proof.$\square$

**Corollary 1.** Suppose $\pi^m > \rho$ and $V(k + 1) > V(k)$ for any $k \geq 1$. Then, for all $n$, under the optimal strategy, the probability of purchase declines along the referral chain: i.e., $\alpha_1^* > \alpha_2^* > \ldots > \alpha_n^*$.

**Proof of Corollary 1.** If $n$ consumers are left in the chain, then the probability of purchase for the first consumer is $\alpha_1^* = \gamma^*(n)$. If the referral for the second consumer is successful, then $n - 1$ consumers are left in the chain, and the probability of purchase for the second consumer is $\alpha_2^* = \gamma^*(n - 1)$. Similarly, $\alpha_k^* = \gamma^*(n - k + 1)$ for all $k = 1, \ldots, n$. The probability of sales declines along the referral chain (i.e. $\alpha_1^* > \alpha_2^* > \ldots > \alpha_n^*$) because $\gamma^*(n) > \gamma^*(n - 1) > \ldots > \gamma^*(1)$ by Proposition 1.$\square$

**Proposition 2.** Suppose $\pi^m > \rho$. Monopoly profit increases in the length of the consumer chain: $V(k) < V(k + 1)$ for all $k \geq 1$.

**Proof of Proposition 2.** Note that $V(k)$ is described in the following manner:

$$V(k) = \sum_{h=1}^{k-1} \left[ (P(\alpha_h^*) - c - \rho) \left( \prod_{\ell=1}^{h} \alpha_\ell^* \right) \right] + P(\alpha_k^*) \prod_{\ell=1}^{k} \alpha_\ell^*, \quad (34)$$
where $\alpha^*_h = \gamma^*(k - h + 1)$ for all $h$. Let’s look at the monopoly problem with $k + 1$ consumers. Consider the following policy for the firm: set the same purchase probabilities for the first $k$ consumers as in the $k$-consumer problem (i.e., $\alpha_h = \alpha^*_h$ for all $h = 1, \ldots, k$) and make the $k$th consumer just willing to make a referral to the last $(k + 1)$th consumer (by setting the expected referral benefit equal to the referral cost: $\alpha^*_{k+1} r^*_{k+1} = \rho$). The monopoly profit under such a policy is

$$
\Pi(\alpha_{k+1}; \alpha^*_1, \ldots, \alpha^*_k) = V(k) + (\alpha_{k+1} P(\alpha_{k+1}) - c - \rho) \left( \prod_{\ell=1}^{k} \alpha^*_\ell \right).
$$

Note that $\prod_{\ell=1}^{k} \alpha^*_\ell$ is the unconditional probability that the $k$th consumer purchases the product, and the firm pays $\rho$ to let her make a referral. Since the maximum $V$ is achieved with $\alpha_{k+1} = \alpha^m$, we have

$$
\tilde{\Pi} = \max_{\alpha_{k+1}} \Pi(\alpha_{k+1}; \alpha^*_1, \ldots, \alpha^*_k) = V(k) + (\alpha^m P(\alpha^m) - c - \rho) \left( \prod_{\ell=1}^{k} \alpha^*_\ell \right) > V(k),
$$

if and only if $\pi^m = \alpha^m (P(\alpha^m) - c) > \rho$. Thus, we have $\tilde{\Pi} > V(k)$. Since $V(k + 1) \geq \tilde{\Pi}$, it follows that $V(k + 1) > V(k)$ whenever $\pi^m > \rho$. \hfill \square

**Theorem 2.** Suppose $\pi^m > \rho$. For all $n$, the firm’s optimal policy satisfies $\alpha^*_1 > \alpha^*_2 > \ldots > \alpha^*_n = \alpha^m$, $p^* = P(\alpha^m)$, and the expected referral benefits for consumers 1 through $n - 1$ satisfy $\alpha^*_2 r^*_2 > \alpha^*_3 r^*_3 > \ldots > \alpha^*_n r^*_n > 0$.

**Proof of Theorem 2.** From Proposition 2 and Corollary 1, we know that $\alpha^*_1 > \alpha^*_2 > \ldots > \alpha^*_n = \alpha^m$ and $p = P(\alpha^*_n) = P(\alpha^m)$. From (8), $\alpha^*_k r^*_k = p - P(\alpha^*_{k-1}) + \rho > 0$ for $k = 2, \ldots, n$, and since $\alpha^*_1 > \alpha^*_2 > \ldots > \alpha^*_n$, we conclude that $\alpha^*_2 r^*_2 > \alpha^*_3 r^*_3 > \ldots > \alpha^*_n r^*_n > 0$. \hfill \square
Appendix C: Jun and Kim’s Model for an Infinite Referral Chain

The monopoly profits with no referrals are $\pi(\alpha) = \alpha (P(\alpha) - c)$. They are maximized at $\alpha = \alpha^m$ such that $\pi'(\alpha^m) = 0$. From (4) and (5), monopoly profits with active consumer referrals in the case of $n \to \infty$ are

$$\Pi = \frac{1}{1-\alpha} (\pi (\alpha) - \rho) + \rho = \frac{\alpha}{1-\alpha} (P(\alpha) - c - \rho),$$

where $\pi (\alpha) = \alpha (P(\alpha) - c)$ and $\alpha = D(p - \alpha r + \rho)$. Solving the last equality for $r$, we find that

$$r = \frac{1}{\alpha} (p - P(\alpha) + \rho)$$

The profit-maximizing probability of purchase is defined by the first-order condition

$$\frac{d\Pi}{d\alpha} = \left( \frac{1}{1-\alpha} (\pi (\alpha) - \rho) \right)' = \frac{1}{(1-\alpha)^2} (\pi (\alpha) - \rho) + \frac{1}{1-\alpha} \pi'(\alpha) = 0. $$

Assuming $\pi^m \equiv \pi(\alpha^m) > \rho$, the first-order condition implies that $\frac{d\Pi}{d\alpha}|_{\alpha=\alpha^m} > 0$, and therefore $\alpha^* > \alpha^m$.

For the uniform $U[0,1]$ distribution of valuations, we can obtain the explicit solution:

$$\alpha^* = 1 - \sqrt{c + \rho}$$

and

$$r^* = \frac{1}{1-\sqrt{c + \rho}} (p - \sqrt{c + \rho} + \rho).$$

Note that $r^*$ is increasing in $p$: a higher price is associated with a higher referral fee.
The optimal monopoly profits with active consumer referrals are \( \Pi^* = (1 - \sqrt{c + \rho})^2 \), while the standard monopoly profits (with no referrals) are \( \pi^m = \frac{1}{4} (1 - c)^2 \). A comparison of profits reveals that \( \Pi^* \geq \pi^m \) if and only if \( \rho \leq \frac{1}{4} (1 - c)^2 \). The referral condition \( \alpha^* r^* - \rho = p - \sqrt{c + \rho} \geq 0 \) is satisfied whenever \( p \geq \sqrt{c + \rho} \). Hence, for any \( \rho \leq \frac{1}{4} (1 - c)^2 \) there is a continuum of optimal strategies \((p^*, r^*)\): \( p^* \geq \sqrt{c + \rho} \) and \( r^* = \frac{\rho}{1 - \sqrt{c + \rho}} (p - \sqrt{c + \rho} + \rho) \) that support referrals. The referral condition is binding at the lowest optimal price and referral fee, \( p^* = \sqrt{c + \rho} \) and \( r^* = \frac{\rho}{1 - \sqrt{c + \rho}} \), and not binding at higher \( p^* \) and \( r^* \). The lowest price \( p^* = \sqrt{c + \rho} \) is below the standard monopoly price \( p^m = \frac{1}{2} (1 + c) \) because \( \sqrt{c + \rho} \leq \frac{1}{2} (1 + c) \) whenever \( \rho \leq \frac{1}{4} (1 - c)^2 \). For the lowest optimal price and referral fee, we have monotonic comparative statics results – the optimal price and referral fee increase in referral and production costs \( \rho \) and \( c \).