Discontinuous Extraction of a Nonrenewable Resource

by

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Abstract
This paper examines the optimal extraction sequence of nonrenewable resources in the presence of multiple demands. We provide conditions under which extraction of a nonrenewable resource may be discontinuous over the course of its depletion.

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1. Introduction

A fundamental theorem of resource economics suggests that extraction of identical deposits of a nonrenewable resource should be in the order of their cost of extraction (e.g., Herfindahl (1967), Solow and Wan (1976), and Lewis (1982)). Gaudet, Moreaux and Salant (2001), use a model of trash hauling between cities (in our case, demands) and landfills with a fixed capacity (analogous to resources), to suggest that in the presence of setup costs a city may temporarily abandon a low marginal cost site (and use a higher cost one) and return to the former at a later date. An implication of their result is that a nonrenewable resource may be extracted discontinuously, i.e., over two disjoint time periods. In this paper, we show that discontinuous extraction of a nonrenewable resource is still possible across demands, even without setup costs. That is, a resource may be used in a particular demand, then abandoned for a time, only to be used in another demand later in time. We provide conditions for this phenomenon to occur.

We modify the framework of dynamic optimization in Chakravorty and Krulce (1994, henceforth CK) who consider two nonrenewable resources, oil ($O$) and coal ($C$), for two demands, electricity ($E$) and transportation ($T$), by adding a third backstop resource ($B$) with an infinite supply (e.g., solar power). While the assumption of a constant unit extraction cost in CK is retained for each resource ($c_{i}$, $i = O, C, B$), we specify conversion costs as both resource and energy specific ($z_{j}$, $i = O, C, B$; $j = E, T$) so that the net cost of resource $i$ for demand $j$ is $w_{ij} = c_{i} + z_{ij}$.

The planner chooses instantaneous extraction rates of resource $i$ for demand $j$, $q_{ij}(t)$, which maximizes the discounted social surplus, $W$:

$$W = \int_{0}^{\infty} e^{-\tau t} \left[ \sum_{j} \left( \int_{0}^{\infty} D_{j}^{-1}(x) dx \right) - \sum_{i,j} (c_{i} + z_{ij}) q_{ij}(t) - \sum_{i} \lambda_{i}(t) \sum_{j} q_{ij}(t) \right] dt$$

subject to

\footnote{At least three resources are needed for discontinuous extraction, which is also the case in Gaudet et al.}
where \( r \) denotes the discount rate, \( D_j^{-1} \) the inverse energy demand function for demand \( j \), \( Q_j(t) \) the initial stock (assumed known) of resource \( i \), and \( \lambda_i(t) \) the co-state variable for resource \( i \). Define the resource price for demand \( j \) as \( p_j(t) = D_j^{-1}(\sum_i q_{ij}(t)) \) and the price of resource \( i \) for demand \( j \) as \( p_{ij}(t) = c_i + z_{ij} + \lambda_i(t) \equiv w_{ij} + \lambda_i(t) \). The necessary and sufficient conditions\(^3\) are

\[
\dot{Q}_j(t) = -\sum_i q_{ij}(t) 
\]

\[
\dot{\lambda}_i(t) = r\lambda_i(t) 
\]

\[
p_j(t) \leq p_{ij}(t) \quad \text{(if } < \text{ then } q_{ij}(t) = 0 \text{)}
\]

\[
\lim_{t \to \infty} e^{-rt} \lambda_i(t) \geq 0; \quad \lim_{t \to \infty} e^{-rt} \lambda_i(t) Q_i(t) = 0
\]

where (2) implies \( \lambda_i(t) = \lambda_i(0) e^{rt} \). Substituting (2) into (4) for nonrenewable resource \( i \) (\( = C, O \)), we obtain \( \lim_{t \to \infty} e^{-rt} (\lambda_i(0) e^{rt}) Q_i(t) = \lambda_i(0) \lim_{t \to \infty} Q_i(t) \) which gives \( \lim_{t \to \infty} Q_i(t) = 0 \).

For the backstop resource \( (i = B) \) which is in infinite supply, \( Q_B(t) > 0 \) for all \( t \), so that \( \lim_{t \to \infty} e^{-rt} (\lambda_B(0) e^{rt}) Q_B(t) = \lambda_B(0) \lim_{t \to \infty} Q_B(t) \) which yields \( \lambda_B(0) = 0 \), hence \( \lambda_B(t) = 0 \).

**2. Optimal Extraction Sequence**

Let us consider the special case in which oil is the cheapest resource for all demands and the backstop is the most expensive:

**Assumption**: \( 0 < w_{Oj} < w_{Gj} < w_{Bj} < \infty \).

\(^3\) The proof of sufficiency is essentially the same as in CK, hence suppressed.
Then, as shown by Chakravorty, Krulce, and Roumasset (2003), the ordering of the shadow prices is exactly the reverse of the ordering of net costs $w_{ij}$, i.e.,

$$\lambda_{oi}(t) > \lambda_{ci}(t) > \lambda_{bi}(t) = 0.$$  

Note that since both oil and coal are nonrenewable resources, they will eventually be exhausted and the backstop will be used for both demands. By Proposition 7 of their paper, the order of extraction in each demand must be in the order of the net costs ($OCB$), i.e., oil followed by coal and then by the backstop (as illustrated in Fig. 1), although not every resource need be extracted for each demand.

### 3. Conditions for Discontinuous Resource Extraction

In this section, we demonstrate graphically the possibility of discontinuous extraction of a nonrenewable resource, and then provide necessary and sufficient conditions for the discontinuity to occur. In Fig.1, the (energy) resource price for each demand is depicted as an envelope curve: in bold solid for transportation and in bold dash for electricity. Note that coal is extracted in phase II, and again in phase IV. There is no extraction of coal in the intermediate phase III.

<Figure 1 here>

The switch point sequence for Fig. 1 is $S1$: $0 < t_{1E} < t_{2E} < t_{1T} < t_{2T} < \infty$. Note that if the $p_{OE}(t)$ curve in the lower part of Fig. 1 shifts up, oil may not be used in electricity, but coal extraction will remain discontinuous but with an altered switch point sequence $S2$: $t_{1E} \leq 0 < t_{2E} < t_{1T} < t_{2T} < \infty$. Either $S1$ or $S2$ is equivalent to the following three inequalities:

\begin{align}
(i). & \quad 0 < t_{2E} \\
(ii). & \quad t_{2E} < t_{1T} \\
(iii). & \quad t_{1j} < t_{2j}, \quad j = E,T. 
\end{align}

(5)
Under these two sequences, coal is extracted first for electricity and then for transportation. We can now state

**PROPOSITION 1:** Coal is extracted discontinuously, first for electricity (E) and then for transportation (T) after a time delay, iff

\[(a). \quad \lambda_c(0) < w_{BE} - w_{CE} ;\]

\[(b). \quad 1 + \max \left\{ \frac{w_{Cj} - w_{Oj}}{w_{Bj} - w_{Cj}} \right\} \frac{\lambda_O(0)}{\lambda_C(0)} < 1 + \frac{w_{CT} - w_{OT}}{w_{BE} - w_{CE}} , \quad j = E, T .\]

**Proof:** At the switch points for demand \(j\), \(p_{oj} = p_{cj}\) and \(p_{cj} = p_{bj}\), which gives:

\[
t_{ij} = \frac{1}{r} \ln \frac{w_{Cj} - w_{Oj}}{\lambda_O(0) - \lambda_C(0)} ; \quad t_{2j} = \frac{1}{r} \ln \frac{w_{Bj} - w_{Cj}}{\lambda_C(0)} . \tag{6}\]

In view of (5), it suffices to show that (i), (ii) and (iii) are equivalent to conditions (a) and (b).

Since \(0 < \lambda_C(0) < \lambda_O(0) < \infty\) in light of the Assumption in Section 2,

\[
(i). \quad 0 < t_{2E} \quad \Leftrightarrow \quad (a). \quad \lambda_C(0) < w_{BE} - w_{CE} ;
\]

\[(ii). \quad t_{2E} < t_{1T} \quad \Leftrightarrow \quad (b). \quad \frac{\lambda_O(0)}{\lambda_C(0)} < 1 + \frac{w_{CT} - w_{OT}}{w_{BE} - w_{CE}} ;\]

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4 There exists a subset of \(w = (w_{OE}, w_{CE}, w_{BE}, w_{OT}, w_{CT}, w_{BT})\) which admits conditions (a) and (b), e.g., \(w = (4, 5, 6, 1, 4, 6)\). Given \(w\), \(\lambda_O(0)\) and \(\lambda_C(0)\) still depend on other factors such as the initial stocks of resources, the discount rate and the magnitude of demands, hence are not determined solely by \(w\).
(iii). $t_{ij} < t_{2j}$ \iff \frac{w_{Cj} - w_{Oj}}{\hat{\lambda}_O(0) - \hat{\lambda}_C(0)} < \frac{w_{Bj} - w_{Cj}}{\hat{\lambda}_C(0)} \iff (b2). \quad \frac{\hat{\lambda}_O(0)}{\hat{\lambda}_C(0)} > 1 + \max \left\{ \frac{w_{Cj} - w_{Oj}}{w_{Bj} - w_{Cj}} \right\}.

Noting that \((b1)\) and \((b2)\) jointly are equivalent to \((b)\) in Proposition 1 completes the proof. \quad Q.E.D.

The conditions in Proposition 1 can be re-expressed as three simple inequality constraints for $\lambda_c(0)$ and $\lambda_o(0)$: $\lambda_c(0) < \alpha$, $\lambda_o(0) < \beta \lambda_c(0)$, and $\lambda_o(0) > \gamma \lambda_c(0)$ where

$$\alpha \equiv w_{BE} - w_{CE} > 0;$$
$$\beta \equiv 1 + \frac{w_{CT} - w_{OT}}{w_{BE} - w_{CE}} > 1; \text{ and}$$
$$\gamma \equiv 1 + \max \left\{ \frac{w_{Cj} - w_{Oj}}{w_{Bj} - w_{Cj}} \right\} > 1,$$

which are graphed in Fig. 2. The entire grey area represents an open set of $\{(\lambda_c(0), \lambda_o(0)) \mid 0 < \lambda_c(0), \lambda_o(0) < \infty\}$ which satisfies the conditions for either $S1$ or $S2$ to occur. The only difference between the two sequences is that $0 < t_{1E}$ for $S1$ and $t_{1E} \leq 0$ for $S2$. Setting $t_{1E} = 0$ in (6), we obtain $\lambda_o(0) = \mu + \lambda_c(0)$ where $\mu \equiv w_{CE} - w_{OE} > 0$. Hence, we can restate the difference in terms of $\lambda_c(0)$ and $\lambda_o(0)$: $\lambda_o(0) > \mu + \lambda_c(0)$ for $S1$ and $\lambda_o(0) < \mu + \lambda_c(0)$ for $S2$. In Fig. 2, the line $\lambda_o(0) = \mu + \lambda_c(0)$ splits the grey area into two. The dark grey area, an open set, is the domain of $(\lambda_c(0), \lambda_o(0))$ which admits sequence $S1$, and the light grey area which is inclusive of the splitting line is the domain of $(\lambda_c(0), \lambda_o(0))$ for sequence $S2$. If $\mu \geq \mu^* \equiv \alpha(\beta - 1)$, there exists no $(\lambda_c(0), \lambda_o(0))$ which admits sequence $S1$, and the entire grey area is the domain for sequence $S2$. 
Note that Fig. 1 is symmetric with respect to $E$ and $T$. That is, if $E$ and $T$ were interchanged, coal would still be extracted discontinuously with the altered switch point sequences $S1': 0 < t_{1T} < t_{2T} < t_{1E} < t_{2E} < \infty$ and $S2': t_{1T} \leq 0 < t_{2T} < t_{1E} < t_{2E} < \infty$ which are identical to $S1$ and $S2$. Proposition 1 will continue to hold in this case. Since \{S1, S2, S1', S2'\} is the complete set of switch point sequences which admits discontinuous coal extraction, we may then generalize Proposition 1 (without proof) as

**PROPOSITION 2 (General):** Coal is extracted discontinuously iff

\[
\begin{align*}
(a) & \quad \lambda_c(0) < w_{Bj} - w_{Cj} ; \\
(b) & \quad 1 + \max \left\{ \frac{w_{Ch} - w_{Oh}}{w_{Bh} - w_{Ch}}, \frac{w_{Cj'} - w_{Oj'}}{w_{ Bj'} - w_{Cj}} \right\} < 1 + \frac{w_{Cj'} - w_{Oj'}}{w_{ Bj'} - w_{Cj}} , \quad (j, j^* = E \text{ or } T; j \neq j^*; h = j, j^*).
\end{align*}
\]

4. Conclusion

This paper provides conditions under which a nonrenewable resource can be extracted discontinuously in a model with two resources and a renewable backstop resource. Given that the Herfindahl principle (least cost first) must be preserved within each demand, a total of three resources are necessary for the discontinuous extraction to occur. In a typical economy with multiple demands for resources with different grades, an observed pattern of discontinuous resource extraction may appear chaotic, but may actually be a result of optimal behavior as suggested in this paper.
References


Fig. 1: Discontinuous Coal Extraction: Coal is extracted in phase II and IV, but not in III.
Fig. 2: Coal Extraction is Discontinuous in the Shaded Open Set