Beauty Contest with Rationally Inattentive Agents

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March 13, 2017

Abstract

In the context of a "beauty-contest" coordination game, players choose how much costly attention to pay to public information. Introducing information costs based on rational inattention implies that, in the neighborhood of zero information costs, multiple equilibria can emerge in settings that without information costs would imply unique linear equilibrium. Agents have a coordination motive arising from strategic complementarity in their actions, which, in turn, implies coordinating on attention devoted to the public signal. This effect induces multiple equilibrium levels of attention at intermediate levels of transparency of public information (worth paying attention to if others do, but not worth paying attention to if others do not) for small enough information costs. Formally, the set of equilibria under rational inattention does not converge to the set of equilibria without information costs as the price of attention approaches zero. Quintessentially, small deviations from rationality can make significant differences to economic equilibria.

*jkim230@emory.edu. I am grateful to Chris Sims, Faruk Gul, Stephen Morris, and Ming Yang for their comments. I also thank participants of the North American Summer Meetings of the Econometric Society and the 2014 Summer School of the Econometric Society in Seoul. This work was partially supported by the Center for Science of Information (CSoI), an NSF Science and Technology Center, under grant agreement CCF-0939370.

1 Introduction

In a wide variety of social settings, decision-makers take actions appropriate to the unknown underlying fundamentals (a "fundamental" motive), but they also have a coordination motive arising from a strategic complementarity in their actions (a "coordination" motive). In such settings, the participants may welcome greater access to information that permits actions better suited to the circumstances.

When people's abilities to translate external data into action are constrained, they must balance the cost of information against its benefit. That benefit depends on two things if the information is public (shared by everyone). On the one hand, it conveys information on the underlying fundamentals, but it may also serve as a focal point for the beliefs of the group as a whole if others pay close attention to it. In a two-stage listening-then-acting environment, this paper asks: how do their information acquisition decisions respond to the properties of their environment, and can small deviations from fully optimal choices have big consequences on the equilibria of a coordination game?

Formally, I introduce information costs based on rational inattention in a model of coordination game that is based on one popular shortcut, introduced by Morris and Shin (2002), to formalize Keynes' beauty contest metaphor for financial markets. Introducing information costs based on rational inattention implies that, in the neighborhood of zero information costs, multiple equilibria can emerge in settings that without information costs would imply unique linear equilibrium. Interestingly, asdf. The contribution of my paper is to establish that small deviations from fully optimal choices arising from costly attention can have big consequences on the equilibria of a "beauty-contest" game.

Surely, it ought to be evident that for sufficiently high information cost, endogenous attention allocation will significantly alter the equilibrium properties. Indeed, we will begin by showing that for sufficiently high information cost, agents never pay attention to the public information and in turn, its welfare effect is null.

Thus, more specifically, we ask the question, can small deviations from fully rationality via rational inattention, or recognizing that attending to information is costly, have significant consequences? In turn, we study the case of small, but positive, cost of attention, i.e., "almost perfect attention" and compare its equilibrium properties to the case of perfect attention (studied by Morris and Shin (2002)). We will see that our answer is affirmative, and the case of "almost perfect attention" is fundamentally different from the case of perfect attention in the sense that the latter is not simply the limiting case of the former. Thus, our results are somewhat reminiscent of the important contribution of Rubinstein (1989), who shows that, in coordination games, "almost common knowledge" is very different from common knowledge.

Quintessentially, the running theme of my results is that action complementarity implies attention complementarity. First, we show that, if the preference for strategic complementarity between actions is strong, for intermediate levels of precisions of the public signal, the private value of paying attention to the public signal hinges on how much attention other agents in the economy are already paying to it. This, in turn, leads to fluctuations in private value of attending to the public signal that we call attention complementarity. And attention complementarity is more susceptible to multiple-equilibria phenomenon (than action complementarity). For low levels of precisions of the public signal, the information costs of attending to it at all do not justify the returns from doing so. This is because no matter how small the cost, for sufficiently noisy public signal, the costs of paying attention will outweigh the benefits. Only when the public signal is sufficiently precise does there exist unique equilibrium in which agents nontrivially react to public information.

Beauty-contest models have received close attention following the contribution of Morris and Shin (2002), who show that increased public disclosures can be detrimental to welfare when agents also have access to independent sources of information. There followed numerous papers considering models with public and private information (Angeletos and Pavan, 2004; Hellwig, 2005).[1] This earlier literature raised interesting questions about the value of "transparency" and arrived at conflicting conclusions, depending on assumptions about externalities that are difficult to calibrate against an actual economy.

But the models from this earlier literature all assume that there are public information sources whose stochastic characters policy makers can controlling. Agents in turn have access to these information sources with exogenously given attributes. From a rational inattention perspective, this is a strange setup. Rationally inattentive agents will act as if observing the state of the economy with error even if some public

[1]Beauty-contest games have been applied to investment games (Angeletos and Pavan, 2004), to monopolistic competition (Hellwig, 2005), to financial markets (Allen, Morris and Shin, 2006), to a range of other economic problems (Angeletos and Pavan, 2007), and to political leadership (Dewan and Myatt, 2008); many other papers report variants of the beauty-contest specification.

authority announces it exactly. Equilibrium properties of the observation error, which is the object of interest for us, will depend both on the stochastic properties of the state itself and of any noise in public signals about it.

I am not alone in trying to understand the impact of endogenous information acquisition in beauty-contest games. Hellwig and Veldkamp (2009) noted first that strategic complementarity (of actions) imposes complementarity in information choices, which coupled with discreteness in choice over multiple public information creates multiple equilibria. Their intuition applies here and to Myatt and Wallace (2012), which is the most closely related research to this paper. Myatt and Wallace assume that the cost of acquiring information is linear in the observation precision, whereas I assume that this function is the Shannon measure of mutual information. The particular cost function they assume suffices for equilibrium uniqueness. However, this is not true with rational inattention. This is because of the concavity of entropy. Intuitively, Shannon entropy implies that the information content arising from additional data is decreasing in the stock of existing information, so the cost of passing on the information rises only concavely. This discussion suggests that implications of endogenous information acquisition is quite sensitive to its formulation.

In strategy games, when players are rationally inattentive, it has been suggested that there is scope for multiple equilibria (Myatt and Wallace, 2012; Yang, 2015). The novel contribution of my paper is to show that, in beauty-contest games, this effect induces multiple equilibrium levels of attention for small enough information costs, so equilibrium properties at small deviations from fully optimal choices are discretely different than fully optimal choices. Importantly, this implies that a public authority facing near-rational agents ought to worry about indeterminacy, something it does not need to worry facing fully rational agents.

More broadly, the study adds to the "rational-inattention" literature associated with Sims (2003) considering endogenous information acquisition. In finance, van Nieuwerburgh and Veldkamp (2004) and Peng and Xiong (2005), for example, have applied information-theoretic ideas.[2] This paper was inspired by Sims (2005), who argued for consideration of how strategic games with public and private information are affected by rational inattention.

[2] See Matějka (2016) and Jung et al. (2016) for applications of rational inattention to models in which objective functions are not necessarily quadratic and supports of distributions are not necessarily unbounded.

Finally, note that the literature without endogenous information acquisition analyzed welfare for games that have externalities, strategic interaction, and heterogeneous information. But for all the diversity in the conclusions from this literature, and for the sensitivity of equilibrium implications of endogenous information acquisition on its formulation, I choose to focus on the implications of rational inattention on equilibrium (rather than welfare) properties in this paper.[3][4]

Turning to the structure of the paper, Sections 2 describes the model and solve for individual agent's optimal action. Section 3 solve for symmetric Nash equilibria and show how information acquisition respond to the coordination motive and to other parameters and, in turn, leads to multiple equilibria with near-rational agents. Finally, Section 4 concludes.

2 Model

Our model is based on one possible shortcut, followed by Morris and Shin (2002), to formalize Keynes' beauty contest metaphor for financial markets. There is a continuum of agents, indexed by the unit interval $[0,1]$. Agent i chooses an action $a_i \in \mathbb{R}$, and we write $\mathbf{a}$ for the action profile over all agents. The payoff function for agent i is given by

$$u_i(\mathbf{a},\theta) \equiv -(1-r)(a_i-\theta)^2 - rL_i - \lambda I(y,y_i)$$

where r is a constant, with $0<r<1$ and

$$L_i \equiv \int_0^1 (a_j - a_i)^2\, dj.$$

The loss function for individual i has three components. As in Morris and Shin (2002), the first component is a standard quadratic loss in the distance between the underlying state θ and his action a_i. The second component is the "beauty contest" term that induces each individual to try to second-guess the decisions of other individuals in the economy. The parameter r gives the weight on this second-guessing motive.

On the other hand, we depart from the model by assuming that individual i must observe the public signal y through a finite-capacity channel, with his observation y_i.

[3]Some preliminary welfare analysis for the model in this paper is available upon request.

[4]More recently, Colombo et al. (2014) and Pavan (2016), respectively, consider welfare in games of strategic interactions with information acquisition and under bounded rationality.

Formally, the third term captures this constraint, where ψ is the cost of information, with $\psi > 0$ and

$$I(y; y_i) = \int \log f(y, y_i) f(y, y_i) \, dy dy_i - \int \log f(y) f(y) \, dy$$

is the mutual information between the two jointly distributed varibles y and y_i. We interpret this expression as the average reduction in entropy when we use observations on y_i to reduce uncertainty about y. Social welfare, defined as the (normalized) average of individual utilities, is

$$\begin{aligned} W(\mathbf{a}, \theta) &\equiv \frac{1}{1-r} \int_0^1 u_i(\mathbf{a}, \theta) \, di \\ &= -\int_0^1 (a_i - \theta)^2 \, di \end{aligned}$$

so that a social planner who cares only about social welfare seeks to keep all agents' actions close to the state θ. From the point of view of agent i, however, his action is determined by the first-order condition taking the mutual information between y and y_i as given:

$$a_i = (1-r) E_i(\theta) + r E_i(\overline{a})$$

where $\overline{a}$ is the average action in the population (i.e., $\overline{a} = \int_0^1 a_j dj$) and $E_i(\cdot)$ is the expectation operator for player i.

Suppose agents face uncertainty concerning θ, but they have access to public and private information about θ. The state θ is drawn from a normal prior with mean μ_θ and variance σ_θ^2. Without loss of generality, assume that $\mu_\theta = 0$ and $\sigma_\theta^2 = 1$. The public authority announces a *public signal*

$$y = \theta + \eta,$$

but capacity-limited agent i will act as if observing the signal with error, which we capture by letting the agent's decision be made after observing

$$y_i = y + \phi_i$$

where the information-processing errors ϕ_i of the continuum population are normally distributed with zero mean and variance σ_i^2 for $i \in [0, 1]$, indepenent of θ and η, so that $E\left(\phi_i \phi_j\right) = 0$ for $i \neq j$.

In addition to the imperfectly observed public signal y_i, agent i observes the realization of a *private signal*:

$$x_i = \theta + \varepsilon_i$$

where noise terms (ϕ_i, ε_i) of the continuum population are normally distributed with zero mean and variances, independent of θ and η, so that $E\left(\varepsilon_i \varepsilon_j\right) = 0$ for $i \neq j$. We also assume that $E\left(\phi_i \varepsilon_j\right) = 0$ for $i, j \in [0, 1]$. This assumption reflects the fact that the agents do not observe the noise in their private signals and that their information processing is private.

In our formulation, we consider agents who have an information processing cost for the public signal and no cost for attending to the private signal. We could motivate by supposing that the private signal is something relevant to an important decision problem at the individual level - e.g., a business owner's own sales, which he will observe very closely, and as a byproduct get a noisy signal about some aggregate macro variable. The technical motivation is to clearly expose the dependence of equilibrium attentiveness to the public signal on how the public authority sets the stochastic character of the public information source.

Denote by

$$a_i\left(\mathcal{I}_i\right)$$

the decision by agent i as a function of his information set $\mathcal{I}_i$. The information set $\mathcal{I}_i$ consists of the pair (y_i, x_i) that captures all the information available to i at the time of decision.

Let us denote by α_i the endogenous precision of the public information that accounts for agent i's information-processing error, and denote by β the precision of the private information, where

$$\alpha_i = \frac{1}{\sigma_\eta^2 + \sigma_i^2}$$
$$\beta = \frac{1}{\sigma_\varepsilon^2}.$$

Then, based on both private and public information, agent i's expected values of θ

and y are

$$E_i(\theta) = \frac{\alpha y_i + \beta x_i}{\alpha + \beta + 1}$$
$$E_i(y) = \frac{\alpha\left(\left(\beta+1\right)\sigma_\eta^2 + 1\right) y_i + \beta\left(1 - \alpha\sigma_\eta^2\right) x_i}{\alpha + \beta + 1}$$

where we have used the shorthand $E_i(\cdot)$ to denote the conditional expectation $E(\cdot|\mathcal{I}_i)$.

2.1 Agent i's Optimal Action

We will now solve for a Nash equilibrium. We do this in two steps. We first solve for agent i's uniquely optimal action which is a linear function of his signals taking individual information-processing error variances as given. We will follow this with solving for agent i's optimal attention allocation to the public signal. After all, we will want to solve for a symmetric Nash equilibrium in which the individual information-processing error variances are the same across all agents, i.e., all agents pay the same attention to the public signal. Thus, as the first step, suppose that the population as a whole, except possibly for agent i, pays the same attention to the public signal, i.e., $\sigma_j^2 = \sigma^2 \geq 0$ for $j \neq i$.

Recall that player i's best response is to set

$$a_i = (1-r) E_i(\theta) + r E_i(\overline{a}).$$

Substituting and writing $\overline{E}(\theta)$ for the average expectation of θ across agents we have

$$\begin{aligned} a_i &= (1-r) E_i(\theta) + r(1-r) E_i\left(\overline{E}(\theta)\right) + r^2(1-r) E_i\left(\overline{E}^2(\theta)\right) + \cdots \\ &= (1-r)\sum_{k=0}^{\infty} r^k E_i\left(\overline{E}^k(\theta)\right). \end{aligned} \tag{1}$$

Recall that the expected value of θ for player $j \neq i$ is

$$E_j(\theta) = \frac{\alpha y_j + \beta x_j}{\alpha + \beta + 1}. \tag{2}$$

Thus, the average expectation of θ across agents is

$$\overline{E}\left(\theta\right)=\int_0^1 E_i\left(\theta\right)di=\frac{\alpha y+\beta\theta}{\alpha+\beta+1}.$$

Now player j's expectation of the average expectation of θ across agents is

$$\begin{aligned}E_j\left(\overline{E}\left(\theta\right)\right) &= E_j\left(\frac{\alpha y+\beta\theta}{\alpha+\beta+1}\right)\\ &= \frac{\alpha\left(\frac{\alpha\left[(\beta+1)\sigma_\eta^2+1\right]y_j+\beta\left(1-\alpha\sigma_\eta^2\right)x_j}{\alpha+\beta+1}\right)+\beta\left(\frac{\alpha y_j+\beta x_j}{\alpha+\beta+1}\right)}{\alpha+\beta+1}\\ &= \frac{\alpha\left(\alpha\left[(\beta+1)\sigma_\eta^2+1\right]+\beta\right)y_j+\beta\left(\alpha\left(1-\alpha\sigma_\eta^2\right)+\beta\right)x_j}{\left(\alpha+\beta+1\right)^2}\end{aligned}$$

and the average expectation of the average expectation of θ is

$$\overline{E}^2\left(\theta\right)=\overline{E}\left(\overline{E}\left(\theta\right)\right)=\frac{\alpha\left(\alpha\left[(\beta+1)\sigma_\eta^2+1\right]+\beta\right)y+\beta\left(\alpha\left(1-\alpha\sigma_\eta^2\right)+\beta\right)\theta}{\left(\alpha+\beta+1\right)^2}.$$

More generally, we have the following lemma.

Lemma 1 *For any k, $\overline{E}^k\left(\theta\right)=P_2P^{k-1}\begin{bmatrix}y\\ \theta\end{bmatrix}$ and $E_j\left(\overline{E}^k\left(\theta\right)\right)=P_2P^k\begin{bmatrix}y_j\\ x_j\end{bmatrix}$ for $j\neq i$ where $P=\frac{1}{\alpha+\beta+1}\begin{bmatrix}\alpha\left((\beta+1)\sigma_\eta^2+1\right) & \beta\left(1-\alpha\sigma_\eta^2\right)\\ \alpha & \beta\end{bmatrix}$ and P_l denote the lth row of the matrix P.*

Proof. The proof is by induction on k. We know from (2) that the lemma holds for $k=1$. Suppose that it holds for $k-1$. Then,

$$E_j\left(\overline{E}^{k-1}\left(\theta\right)\right)=P_2P^{k-1}\begin{bmatrix}y_j\\ x_j\end{bmatrix}$$

so

$$\overline{E}^k\left(\theta\right)=P_2P^{k-1}\begin{bmatrix}y\\ \theta\end{bmatrix}$$

and

$$E_j\left(\overline{E}^k\left(\theta\right)\right)=P_2P^{k-1}\begin{bmatrix}\frac{\alpha\left((\beta+1)\sigma_\eta^2+1\right)y_j+\beta\left(1-\alpha\sigma_\eta^2\right)x_j}{\alpha+\beta+1}\\ \frac{\alpha y_j+\beta x_j}{\alpha+\beta+1}\end{bmatrix}=P_2P^k\begin{bmatrix}y_j\\ x_j\end{bmatrix}$$

which proves Lemma 1. ■

Now substituting the expression from Lemma 1 into equation (1), we obtain

$$\begin{aligned} a_i &= (1-r)\left(E_i(\theta) + r\sum_{k=0}^{\infty} r^k P_2 P^k \left[\begin{array}{c} E_i(y) \\ E_i(\theta) \end{array}\right]\right) \\ &= (1-r)\left(E_i(\theta) + rP_2 (I - rP)^{-1} \left[\begin{array}{c} E_i(y) \\ E_i(\theta) \end{array}\right]\right) \\ &= (1-r)\left(\frac{\alpha_i y_i + \beta x_i}{\alpha_i + \beta + 1} + rP_2 (I - rP)^{-1} \left[\begin{array}{c} \frac{\alpha_i\left((\beta+1)\sigma_\eta^2+1\right)y_i + \beta\left(1-\alpha_i\sigma_\eta^2\right)x_i}{\alpha_i+\beta+1} \\ \frac{\alpha_i y_i + \beta x_i}{\alpha_i+\beta+1} \end{array}\right]\right) \\ &= (1-r)\frac{\alpha_i(1+\alpha+\beta)y_i + \beta\left(\left(1-r\alpha\sigma_\eta^2\right)(1+\beta) + \alpha\left(1-r\alpha_i\sigma_\eta^2\right)\right)x_i}{(\alpha_i+\beta+1)\left[1-r\alpha\sigma_\eta^2 + (1-r)\left(\alpha + \left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]} \end{aligned}$$

This is exactly the unique optimal action player i chooses taking individual information-processing error variances as given.

3 Symmetric Nash Equilibrium

From the solution for a_i, we can solve for the optimal action in terms of the basic random variables θ, η, and $\{\varepsilon_i, \phi_i\}$.

$$\begin{aligned} a_i &= (1-r)\left((\kappa_i + \lambda_i)\theta + \kappa_i(\eta + \phi_i) + \lambda_i\varepsilon_i\right) \\ &= (1-r)\left\{\frac{\left(\alpha_i + \beta\left(1-r\alpha\sigma_\eta^2\right)\right)(1+\beta) + \left(\alpha_i + \beta\left(1-r\alpha_i\sigma_\eta^2\right)\right)\alpha}{(\alpha_i+\beta+1)\left[1-r\alpha\sigma_\eta^2 + (1-r)\left(\alpha + \left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]}\theta \right. \\ &\qquad \left. + \frac{\alpha_i(1+\alpha+\beta)(\eta+\phi_i) + \beta\left(\left(1-r\alpha\sigma_\eta^2\right)(1+\beta) + \alpha\left(1-r\alpha_i\sigma_\eta^2\right)\right)\varepsilon_i}{(\alpha_i+\beta+1)\left[1-r\alpha\sigma_\eta^2 + (1-r)\left(\alpha + \left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]}\right\} \end{aligned}$$

where

$$\begin{aligned} \kappa_i &= \frac{\alpha_i(1+\alpha+\beta)}{(\alpha_i+\beta+1)\left[1-r\alpha\sigma_\eta^2 + (1-r)\left(\alpha + \left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]} \\ \lambda_i &= \frac{\beta\left(\left(1-r\alpha\sigma_\eta^2\right)(1+\beta) + \alpha\left(1-r\alpha_i\sigma_\eta^2\right)\right)}{(\alpha_i+\beta+1)\left[1-r\alpha\sigma_\eta^2 + (1-r)\left(\alpha + \left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]}. \end{aligned}$$

Then

$$
\begin{aligned}
E\left((a_i-\theta)^2\right) &= \operatorname{var}(a_i-\theta) \\
&= ((1-r)(\kappa_i+\lambda_i)-1)^2+(1-r)^2\left(\left(\kappa_i^2/\alpha_i\right)+\left(\lambda_i^2/\beta\right)\right).
\end{aligned}
$$

We can compute the expected loss $E\left(L_i-\overline{L}\right)$ by appealing to the law of large numbers.

$$
\begin{aligned}
L_i-\overline{L} &= ((1-r)(\kappa y+\lambda\theta)-a_i)^2-(1-r)^2\int_0^1\left(\kappa\phi_j+\lambda\varepsilon_j\right)^2 dj \\
&= ((1-r)(\kappa y+\lambda\theta)-a_i)^2-(1-r)^2\left(\kappa^2\left(\frac{1}{\alpha}-\sigma_\eta^2\right)+\left(\lambda^2/\beta\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
E\left(L_i-\overline{L}\right) &= \operatorname{var}((1-r)(\kappa y+\lambda\theta)-a_i)-(1-r)^2\left(\kappa^2\left(\frac{1}{\alpha}-\sigma_\eta^2\right)+\left(\lambda^2/\beta\right)\right) \\
&= (1-r)^2\left[(\kappa+\lambda-(\kappa_i+\lambda_i))^2+2\kappa(\kappa-\kappa_i)\sigma_\eta^2+\left(\kappa_i^2/\alpha_i\right)-\left(\kappa^2/\alpha\right)+\left(\lambda_i^2-\lambda^2\right)/\beta\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa &= \frac{\alpha}{1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)}, \\
\lambda &= \frac{\beta\left(1-r\alpha\sigma_\eta^2\right)}{1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)}.
\end{aligned}
$$

The expected payoff of agent i at the time of attention allocation decision is then given by the expectation:

$$
\begin{aligned}
\frac{E(u_i)(\alpha_i,\alpha)}{(1-r)^2} &= -\frac{\psi}{(1-r)^2}I(y,y_i)-(1-r)((\kappa_i+\lambda_i)-1/(1-r))^2-\left(\left(\kappa_i^2/\alpha_i\right)+\left(\lambda_i^2/\beta\right)\right) \\
&\quad -r\left[(\kappa+\lambda-(\kappa_i+\lambda_i))^2+2\kappa(\kappa-\kappa_i)\sigma_\eta^2-\left(\left(\kappa^2/\alpha\right)+\left(\lambda^2/\beta\right)\right)\right] \\
&= -\frac{\psi}{2(1-r)^2}\log\left(\frac{\alpha_i+1}{1-\alpha_i\sigma_\eta^2}\right)-\frac{1}{1-r}\frac{(1-r\alpha\eta)^2+\left(\alpha+(1-r\alpha\eta)^2\beta\right)(1-r)^2}{\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2} \\
&\quad -\frac{(1+\alpha+\beta)(\alpha-\alpha_i)}{(1+\alpha_i+\beta)\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2}. \qquad (3)
\end{aligned}
$$

Agent i's problem is to choose α_i to maximize expression (3) subject to

$$0 \leq \alpha_i \leq 1/\sigma_\eta^2.$$

We use:

Definition 2 *A symmetric Nash equilibrium* α *satisfies*

1. $\alpha \in \left[0, 1/\sigma_\eta^2\right]$*;*
2. $E\left(u_i\right)\left(\alpha, \alpha\right) = \max_{\alpha_i \in \left[0,1/\sigma_\eta^2\right]} E\left(u_i\right)\left(\alpha_i, \alpha\right)$.

Condition (1) asserts that the precision is by definition at least 0 and is at most $1/\sigma_\eta^2$ if the individual adds no information-processing noise. Condition (2) asserts that assuming

The derivative of $E\left(u_i\right)\left(\alpha_i, \alpha\right)$ with respect to α_i is positive (negative) if and only if

$$\varphi^\alpha\left(\alpha_i\right) = \varphi_0 + \varphi_1 \alpha_i + \varphi_2 \alpha_i^2 \geq (\leq)\, 0$$

where

$$\begin{aligned}
\varphi_2\left(\alpha\right) &= -2\left(1-r\right)^2\left(1+\alpha+\beta\right)^2\sigma_\eta^2 \\
&\quad - \left(1+\sigma_\eta^2\right)\left[1 - r\alpha\sigma_\eta^2 + \left(1-r\right)\left(\alpha + \left(1 - r\alpha\sigma_\eta^2\right)\beta\right)\right]^2\psi \qquad (4) \\
&\quad < 0 \\
\varphi_1\left(\alpha\right) &= 2\left(1-r\right)^2\left(1+\alpha+\beta\right)^2\left(1-\sigma_\eta^2\right) \\
&\quad - 2\left(1+\beta\right)\left(1+\sigma_\eta^2\right)\left[1 - r\alpha\sigma_\eta^2 + \left(1-r\right)\left(\alpha + \left(1 - r\alpha\sigma_\eta^2\right)\beta\right)\right]^2\psi \qquad (5) \\
\varphi_0\left(\alpha\right) &= 2\left(1-r\right)^2\left(1+\alpha+\beta\right)^2 \\
&\quad - \left(1+\beta\right)^2\left(1+\sigma_\eta^2\right)\left[1 - r\alpha\sigma_\eta^2 + \left(1-r\right)\left(\alpha + \left(1 - r\alpha\sigma_\eta^2\right)\beta\right)\right]^2\psi. \qquad (6)
\end{aligned}$$

Let the discriminant of a quadratic equation $p\left(x\right) = 0$ be denoted $D\left(p, x\right)$. For example,

$$\begin{aligned}
D\left(\varphi^\alpha, \alpha_i\right)\left(\alpha\right) &\propto \left(1-r\right)^2\left(1+\alpha+\beta\right)^2\left(1+\sigma_\eta^2\right) \\
&\quad - 2\beta\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)\left[1 - r\alpha\sigma_\eta^2 + \left(1-r\right)\left(\alpha + \left(1 - r\alpha\sigma_\eta^2\right)\beta\right)\right]^2\psi.
\end{aligned}$$

For a quadratic equation that has $D(p,x) > 0$, it will have two real roots. We shall therefore adopt the convention that $x_1(p)$ denotes the larger zero of the polynomial and, when it exists, $x_2(p)$ the smaller zero.

It is useful briefly to describe agent i's optimal policy for attention allocation by studying the polynomial $\varphi^\alpha(\alpha_i)$. Note that, if $D(\varphi^\alpha, \alpha_i) \leq 0$, $\varphi^\alpha \leq 0$ holds with strict inequality for all $\alpha_i \geq 0$ but with equality for at most one point, which follows from the fact that $\varphi_2 < 0$. Similarly, if $D(\varphi^\alpha, \alpha_i) > 0$, $\varphi_1 < 0$ and $\varphi_0 \leq 0$ would imply $\varphi^\alpha \leq 0$ holds with strict inequality for all $\alpha_i \geq 0$ but with equality for at most $\alpha_i = 0$. In particular, α_i is optimally set equal to zero.

We have established that $D(\varphi^\alpha, \alpha_i) > 0$ is necessary for the optimality of a stritly positive α_i. In particular, when $D(\varphi^\alpha, \alpha_i) > 0$, $\varphi_0 > 0$, or $\varphi_1 > \varphi_0 = 0$, and the fact that $\varphi_2 < 0$ imply that agent i attains maximum expected payoff by choosing $\alpha_i = \alpha_1(\varphi^\alpha) > 0$.

Unfortunately, the first-order condition is not sufficient given $D(\varphi^\alpha, \alpha_i) > 0$ and $\varphi_1 > 0 > \varphi_0$. In that case, agent i chooses $\alpha_i = 0$ or $\alpha_1(\varphi^\alpha)$, whichever yields higher expected utility. This completes the best response analysis of agent i taking as given other agents' attention allocation.

It is then straightforward to characterize, under what circumstances a symmetric Nash equilibrium with $\alpha = 0$ exists.

Lemma 3 *Given $D(\varphi^0, \alpha_i)(0) \leq 0$, the model exhibits a symmetric Nash equilibrium with $\alpha = 0$. For $D(\varphi^0, \alpha_i)(0) > 0$, there is a symmetric Nash equilibrium with $\alpha = 0$ if $\varphi_1(0), \varphi_0(0) \leq 0$. Otherwise, there is a symmetric Nash equilibrium with $\alpha = 0$ if and only if $\varphi_1(0) > 0 > \varphi_0(0)$ and $E(u_i)(0,0) \geq E(u_i)(\alpha_1(\varphi^0), 0)$.*

Proof. Recall that by definition, $\alpha = 0$ is a symmetric Nash equilibrium if and only if $0 = \arg\max_{\alpha_i \in [0, 1/\sigma_\eta^2]} E(u_i)(\alpha_i, 0)$, which was established above by using agent i's first-order condition given $D(\varphi^0, \alpha_i)(0) \leq 0$, or $D(\varphi^0, \alpha_i)(0) \leq 0$ and $\varphi_1(0), \varphi_0(0) \leq 0$. Otherwise, the endogenous precision 0 is individually optimal and can be attained in equilibrium only if $\varphi_1(0) > 0 > \varphi_0(0)$. In that case, the first-order condition does not suffice and $\alpha = 0$ is a symmetric Nash equilibrium if and only if $E(u_i)(0,0) \geq E(u_i)(\alpha_1(\varphi^0), 0)$. ■

Notice that $\alpha > 0$ is a symmetric Nash equilibrium only if $\alpha = \alpha_{i1}(\varphi^{\alpha})$, or

$$\begin{aligned} \varphi_0 + \varphi_1 \alpha + \varphi_2 \alpha^2 &= 0, \\ \varphi_1 + 2\varphi_2 \alpha &< 0. \end{aligned}$$

This system is equivalent to

$$\widetilde{\varphi}(\alpha) = \widetilde{\varphi}_0 + \widetilde{\varphi}_1 \alpha + \widetilde{\varphi}_2 \alpha^2 = 0 \tag{7}$$

$$\widehat{\varphi}(\alpha) = \widehat{\varphi}_0 + \widehat{\varphi}_1 \alpha + \widetilde{\varphi}_2 \alpha^2 < 0, \tag{8}$$

where

$$\begin{aligned} \widetilde{\varphi}_2\left(\sigma_\eta^2\right) &= -2(1-r)^2 \sigma_\eta^2 - \left(1+\sigma_\eta^2\right)\left[r(1+(1-r)\beta)\sigma_\eta^2 - (1-r)\right]^2 \psi < 0 \\ \widetilde{\varphi}_1\left(\sigma_\eta^2\right) &= -2(1-r)^2\left(\sigma_\eta^2 - 1\right) \\ &\quad + 2(1+(1-r)\beta)\left(1+\sigma_\eta^2\right)\left[r(1+(1-r)\beta)\sigma_\eta^2 - (1-r)\right]\psi \\ \widetilde{\varphi}_0\left(\sigma_\eta^2\right) &= 2(1-r)^2 - (1+(1-r)\beta)^2\left(1+\sigma_\eta^2\right)\psi \end{aligned} \tag{9}$$

and

$$\widehat{\varphi}_1\left(\sigma_\eta^2\right) = \widetilde{\varphi}_1 - (1-r)^2\left(1+\sigma_\eta^2 + 2\beta\sigma_\eta^2\right) < \widetilde{\varphi}_1 \tag{10}$$

$$\widehat{\varphi}_0\left(\sigma_\eta^2\right) = \widetilde{\varphi}_0 - (1-r)^2\left(1+\beta\left(\sigma_\eta^2 - 1\right) + \sigma_\eta^2\right), \tag{11}$$

because $\varphi^{\alpha}(\alpha) = (1+\alpha+\beta)^2\,\widetilde{\varphi}(\alpha)$ and $\varphi^{\alpha\prime}(\alpha) = 2(1+\alpha+\beta)\,\widehat{\varphi}(\alpha)$. Thus, $D(\widetilde{\varphi}, \alpha) \geq 0$ is necessary for the existence of a symmetric Nash equilibrium with $\alpha > 0$. We can then solve (7)-(8) for $\alpha > 0$ assuming that such a positive number exists. In particular, the equation $\varphi^{\alpha}(\alpha_i) = 0$ satisfies trivially $D(\varphi^{\alpha}, \alpha_i) > 0$, since (7)-(8) implies that there are two real zeros of $\varphi^{\alpha}(\alpha_i)$. And if $\varphi_0(\alpha) \geq 0$, $\alpha > 0$ is a symmetric Nash equilibrium, since $\varphi^{\alpha}(\alpha) = \varphi_0(\alpha) = 0$ would imply that $\varphi_1(\alpha) > 0$. If instead $\varphi_0(\alpha) < 0$, the first-order condition does once again not suffice and $\alpha > 0$ must also satisfy $E(u_i)(\alpha, \alpha) \geq E(u_i)(0, \alpha)$. We summarize this analysis in the following lemma:

Lemma 4 *$D(\widetilde{\varphi}, \alpha) \geq 0$ is necessary for the existence of a symmetric Nash equilibrium with α. Given $D(\widetilde{\varphi}, \alpha) \geq 0$, there is a symmetric Nash equilibrium with $\alpha > 0$ only if it satisfies (7)-(8). If $\varphi_0(\alpha) \geq 0$, $\alpha > 0$ is a symmetric Nash equilibrium.*

Otherwise, it must also satisfy $E(u_i)(\alpha,\alpha) \geq E(u_i)(0,\alpha)$.

We first establish that, for sufficiently large $\psi > 0$, there is a unique symmetric Nash equilibrium such that $\alpha = 0$. That is, each individual chooses to pay no attention to the public signal when information costs are high enough, regardless of the exogenous precision of the public information.

Proposition 5 *Given $r \in (0,1)$ and $\beta \geq 0$, there exists a lower bound $\underline{\psi}(r,\beta)$ on ψ above which the unique symmetric Nash equilibrium is $\alpha = 0$.*

With high enough information processing costs, the proposition shows that, regardless of the exogenous precision of the public information $(1/\sigma_\eta^2)$ or the degree of complementarity (r), it is optimal to set $\alpha = 0$ and so pay no attention to the public signal. A consequence of this proposition is that public information has no impact on welfare in the presence of high information costs. Of course, it is another matter whether agents are really subject to such a tight capacity constraint that the proposition is realistically relevant. Indeed, it is most likely that public information has important welfare effects. But this proposition is a stark illustration of the difference between the environments with fully rational agents (the case in which individuals take as given the stochastic character of public information) and rationally inattentive agents (the case in which individuals endogenize the stochastic character of public information). In the first case, the welfare effects of public information are never trivial.

The key assumption in the preceding proposition is that the information cost parameter ψ is high enough. An interesting question then arises: can injecting a little bit of rational inattention to the benchmark frictionless environment change substantially the nature of equilibrium in the beauty contest model? That is, can the equilibrium behavior of such models postulating agents with small, yet positive costs of capacity differ substantially from that postulating unboundedly rational agents (i.e., $\psi = 0$). We will see that the answer is affirmative for the following reason.

Agents always want more information. Naturally, when $\psi = 0$, therefore the uniquely optimal behavior is to set $\alpha = 1/\sigma_\eta^2$, that is, do full attention. On the other hand, as long as $\psi > 0$, information cost exists and the value of attending to public information depends on how much attention others are paying to this information source. If the exogenous precision of the public signal is such that it is not unambiguously good to pay attention or to simply ignore, this value complementarity will be important. And because the cost of information is very low, but positive, it is possible to accomodate attention levels to match others' behaviors. And of course, complementarity degree will also be very important.

To this end, we start with the proof of the following simple technical lemma. For given $\psi > 0$, define $\underline{\sigma}_\eta^2$ and $\overline{\sigma}_\eta^2$ implicitly by means of

$$\widetilde{\varphi}_0\left(\underline{\sigma}_\eta^2, \psi\right) = 0, \tag{12}$$

$$D\left(\widetilde{\varphi}, \alpha\right)\left(\overline{\sigma}_\eta^2, \psi\right) = 0. \tag{13}$$

Lemma 6 *Suppose* $\psi \leq \frac{2(1-r)^2}{(1+(1-r)\beta)^2}$. *Then*

1. $\widetilde{\varphi}_0\left(\sigma_\eta^2, \psi\right) \leq 0\,(\geq 0)$ *when* $\sigma_\eta^2 \geq (\leq)\,\underline{\sigma}_\eta^2$;
2. $D\left(\widetilde{\varphi}, \alpha\right)\left(\sigma_\eta^2, \psi\right) \leq 0\,(\geq 0)$ *when* $\sigma_\eta^2 \geq (\leq)\,\overline{\sigma}_\eta^2$;
3. $\underline{\sigma}_\eta^2 \leq \overline{\sigma}_\eta^2$, *with equality only if* $r \in \left(0, \frac{1}{2}\right), \beta > 0$ *and* $\psi = \frac{1-2r}{\beta(1+(1-r)\beta)}$, *or* $r = \frac{1}{2}, \beta = 0$.

The following proposition establishes that, for sufficiently small $\psi > 0$, there is a cutoff value of σ_η^2 above which there exists a symmetric Nash equilibrium with $\alpha = 0$. That is, each individual chooses to pay no attention to the public signal (1) when it is noisy enough that if other individuals ignore the public signal, he also optimally ignores the public signal, or (2) when it is just so noisy that regardless of other individuals' attention allocation, the benefits of reacting to the public signal is too slight and in turn he ignores it.

Proposition 7 *Given* $r \in (0,1)$ *and* $\beta \geq 0$, *there exists an upper bound* $\psi\left(r, \beta\right)$ *on* ψ *below which there is a symmetric Nash equilibrium with* $\alpha = 0$ *for* $\sigma_\eta^2 \geq \underline{\sigma}_\eta^2$.

We now establish the following two results. For sufficiently small $\psi > 0$, by reducing the noise in the public signal below $\underline{\sigma}_\eta^2$, the only symmetric Nash equilibrium is such that $\alpha > 0$. That is, each individual chooses to pay attention to the public signal when it is informative enough that regardless of other individuals' attention allocation, the benefits of reacting to the public signal justifies the information costs of attending to it and in turn he never ignores it. With such an informative public signal, the model does not support $\alpha = 0$ as a symmetric Nash equilibrium. On the other hand, for sufficiently small $\psi > 0$, by increasing the noise in the public signal above $\overline{\sigma}_{\eta}^{2}$, the only symmetric Nash equilibrium is such that $\alpha = 0$. We proved in the preceding proposition that there is a symmetric Nash equilibrium with $\alpha = 0$. The present case corresponds to the situation where the public signal is just so noisy that everyone optimally ignore it. With such a noisy public signal, the model does not support $\alpha > 0$ as a symmetric Nash equilibrium. We state these points in the following proposition:

Proposition 8 *Given $r \in (0,1)$ and $\beta \geq 0$, there exists an upper bound $\overline{\psi}(r,\beta)$ on ψ below which there is a unique symmetric Nash equilibrium for $\sigma_{\eta}^{2} \notin \left[\underline{\sigma}_{\eta}^{2}, \overline{\sigma}_{\eta}^{2}\right]$; it is $\alpha = \alpha_1(\widetilde{\varphi}) > 0$ for $\sigma_{\eta}^{2} < \underline{\sigma}_{\eta}^{2}$ and $\alpha = 0$ for $\sigma_{\eta}^{2} > \overline{\sigma}_{\eta}^{2}$.*

It remains to determine the properties of symmetric Nash equilibrium at intermediate levels of precisions of the public signal, i.e., $\sigma_{\eta}^{2} \in \left[\underline{\sigma}_{\eta}^{2}, \overline{\sigma}_{\eta}^{2}\right]$. But in this case, we will see that equilibrium outcomes depends critically on the degree of complementarity, that is, the parameter r. Recall that low r means agents want to match the fundamental relatively more than they want to coordinate their actions. In this case, it is never possible to convince agents to pay attention to the public signal. The unique equilibrium turns out to be what we already figured, a symmetric Nash equilibrium in which everyone simply ignores the public signal. The following proposition establishes this.

Proposition 9 *Given $r \in \left(0, \frac{1}{2}\right)$ and $\beta \geq 0$, there exists an upper bound $\psi(r,\beta)$ on ψ below which there is a unique symmetric Nash equilibrium for $\sigma_{\eta}^{2} \in \left[\underline{\sigma}_{\eta}^{2}, \overline{\sigma}_{\eta}^{2}\right]$; it is $\alpha = 0$.*

Finally, we complete the characterization of equilibrium for near-rationality by considering the case $r \in \left[\frac{1}{2}, 1\right)$. For moderate exogenous precisions of the public signal, there are either two or three equilibria. Of these equilibria, one is always $\alpha = 0$. That is, there is always an equilibrium with everyone ignoring the public signal. Then there may be high attention versus low attention equilibrium. Intuition for this is as follows. Since exogenous precision of the public signal is not too extreme, that is, it is not informative enough that each agent wants to pay attention irrespective of others' attention allocation, and it is not noisy enough that each agent wants to pay no attention irrespective of others' attention allocation, the degree of attention others pay to the public signal is a key determinant of whether one wants to pay attention himself. Of course, if $r < \frac{1}{2}$, the strategic complementarity is weak and the benefits of such attention coordination is small, so positive attention equilibrium cannot be sustained. But with $r \geq \frac{1}{2}$, the strategic complementarity is strong enough to make the benefits of attention coordination privately desirable. Ironically, this is helped by the fact that the information cost is low!

Proposition 10 *Suppose $r \geq \frac{1}{2}$ and $\beta \geq 0$, with at least one of these weak inequalities holding with strict inequality. Then there exists an upper bound $\psi(r, \beta)$ on ψ below which*

1. *there are three symmetric Nash equilibria for $\sigma_\eta^2 \in \left(\underline{\sigma}_\eta^2, \overline{\sigma}_\eta^2\right)$, which are $\alpha = 0$ and $\alpha = \alpha_1(\widetilde{\varphi}), \alpha_2(\widetilde{\varphi}) > 0$;*
2. *there are two symmetric Nash equilibria for $\sigma_\eta^2 = \underline{\sigma}_\eta^2$, which are $\alpha = 0 = \alpha_2(\widetilde{\varphi})$ and $\alpha = \alpha_1(\widetilde{\varphi}) > 0$;*
3. *there are two symmetric Nash equilibria for $\sigma_\eta^2 = \overline{\sigma}_\eta^2$, which are $\alpha = 0$ and $\alpha = \alpha_1(\widetilde{\varphi}) = \alpha_2(\widetilde{\varphi}) > 0$.*

We will end with the case $r = \frac{1}{2}$ and $\beta = 0$. In this case, when $\psi > 0$ is sufficiently small, from Lemma 6 it follows that $\underline{\sigma}_\eta^2 = \overline{\sigma}_\eta^2$. Our Propositions 7 and 8 then shows that the following proposition is true.

Proposition 11 *Given $r = \frac{1}{2}$ and $\beta = 0$, there exists an upper bound $\overline{\psi}(r, \beta)$ on ψ below which there is a unique symmetric Nash equilibrium; it is $\alpha = \alpha_1(\widetilde{\varphi}) > 0$ for $\sigma_\eta^2 < \underline{\sigma}_\eta^2$ and $\alpha = 0$ for $\sigma_\eta^2 \geq \underline{\sigma}_\eta^2$.*

Proposition 11 shows that equilibrium results analogous to the case of $r < \frac{1}{2}$ and $\beta \geq 0$ also characterize the borderline case $r = \frac{1}{2}$ and $\beta = 0$. In general, access to more precise private information increases the length of the interval $\left[\underline{\sigma}_\eta^2, \overline{\sigma}_\eta^2\right]$ where strategic complementarity in attention allocations is most important in determining equilibrium attention to the public signal. Because private information decreases the private value of public information as a source of information on the underlying fundamentals, it renders the coordination motive dominant for an individual to pay attention to it for a larger range of intermediate levels of its exogenous precision. For the case in which there is no private information (i.e., $\beta = 0$), a positive (negative) perturbation away from $r = \frac{1}{2}$ means that the coordination (fundamental) motive is dominant. The effect is unambiguously to increase the region $\left[\underline{\sigma}_\eta^2, \overline{\sigma}_\eta^2\right]$, since the cutoff level $\underline{\sigma}_{\eta}^{2}$ pushed down relatively more than $\overline{\sigma}_{\eta}^{2}$ ($\underline{\sigma}_{\eta}^{2}$ pulled up relative less than $\overline{\sigma}_{\eta}^{2}$) in response to dominance of the coordination (fundamental) motive. In particular, the case $r = \frac{1}{2}$ balances these two motives and the special region vanishes.

3.1 Robustness of Multiple Equilibria

The key point of our analysis is that "almost perfect attention" is very different from when the coordination game is played among agents with perfect attention. This claim follows from the following lemma, whose proof is relatively straightforward and so omitted:

Lemma 12 *As $\psi \to 0$ (i.e., the cost of information approaches zero),*

1. $\underline{\sigma}_{\eta}^{2} \to \infty, \left(\overline{\sigma}_{\eta}^{2} - \underline{\sigma}_{\eta}^{2}\right) \to \infty$*; and*
2. $\underline{\sigma}_{\eta}^{2} / \left(\overline{\sigma}_{\eta}^{2} - \underline{\sigma}_{\eta}^{2}\right) \to const. > 0$*; and*
3. $\alpha_1 (\widetilde{\varphi}) \to 1/\sigma_{\eta}^{2}$.

Clearly, in the case of perfect attention, the unique equilibrium dictates $\alpha = 1/\sigma_{\eta}^{2}$. The last part of Lemma 12 says that, as $\psi \to 0$, the equilibrium, with the attention allocation equal to $\alpha_1 (\widetilde{\varphi})$, converges to this unique equilibrium. But then, in the case of $r > 1/2$, two other distinct equilibria survive in the limit as well.

My main result is that as $\psi \to 0$, there remain three equilibria, even for small ψ, one of which approaches the zero-information cost equilibrium. These equilibria in any sense do not get "closer" to each other as $\psi \to 0$, and the three equilibria remain quite distinct, but that the two that do not approach the full-information equilibrium only occur at levels of public-signal precision that approach zero.

With a fixed σ_{η}^{2}, as $\psi \to 0$, the equilibrium converges to the full-information equilibrium in this scenario. While it remains true that multiple equilibria are possible, the level of imprecision of the public signal required to generate multiple equilibria or $\alpha = 0$ (i.e., "no attention" equilibrium) gets higher and higher as $\psi \to 0$ (with $r > \frac{1}{2}$).

On the other hand, there is robust distinction between the model of Morris and Shin (2002) and our model with the cost of information approaching zero in the sense that $\left(\overline{\sigma}_{\eta}^{2} - \underline{\sigma}_{\eta}^{2}\right) \to \infty$ and $\underline{\sigma}_{\eta}^{2} / \left(\overline{\sigma}_{\eta}^{2} - \underline{\sigma}_{\eta}^{2}\right) \to const. > 0$. This says that, in the case of strong complementarity, the region of multiple equilibria expand to account for a positive share of the positive real line as $\psi \to 0$, so that the former is not simply the limiting case of the latter. Indeed, it is possible to state this non-convergence more formally. For example, in the case of strong complementarity, the number of equilibria does not converge in measure. Of course, if the other two equilibria did not exist or do not survive, the number of equilibria would have converged and in turn, the "almost perfect attention" case would have nicely converged to the case of perfect attention.

All in all, we conclude that multiple equilibria is a robust distinction between the "almost perfect attention" case and the case of perfect attention.

4 Conclusion

In this paper, we have shown that small deviations from the fully rational "beauty contest" game in Morris and Shin (2002) à la rational inattention can have significant implications for the equilibrium properties, and such deviations raise novel issues of efficiency regarding attention allocation. If the preference for action complementarity is strong, for intermediate levels of precisions of the public signal, multiple equilibria, indexed by the level of attention allocation, emerge. More generally, equilibrium attention allocation is inefficient, in the sense that the social planner, if he could, would force agents in the economy to pay more or less attention to the public announcement than they do in equilibrium. The unifying key to these conclusions is that action complementarity translates into attention complementarity.

To stress the role played by attention allocation to the public signal, our model has been deliberately simple. More specifically, we only allow agents to add Gaussian white noise to the public signal. This assumption, while tractable, surely limits the exploration of the implications of rational inattention in coordination games. Thus, an important consideration for future research is to study the existence of other types of equilibrium in this class of models. Understanding optimal policy in such models that endogenize information acquisition would also be fruitful.

5 Appendix

5.1 Proof of Proposition 5:

We have already shown that $D(\varphi^0, \alpha_i)(0) \leq 0$ is sufficient to support $\alpha = 0$ as a symmetric Nash equilibrium, and that $D(\varphi^\alpha, \alpha_i)(0) > 0$ is necessary to support $\alpha > 0$ as a symmetric Nash equilibrium. Then it suffices to show that there exists a $\underline{\psi}(r, \beta) > 0$ such that for all $\sigma_\eta^2 \geq 0$,

$$\psi \geq \underline{\psi}(r, \beta) \implies D(\varphi^\alpha, \alpha_i)(\alpha) \leq 0 \qquad \forall \alpha \in \left[0, 1/\sigma_\eta^2\right].$$

Note that $D(\varphi^\alpha, \alpha_i)(\alpha) \leq 0$ if and only if

$$\psi \geq \frac{(1-r)^2 (1+\alpha+\beta)^2 \left(1+\sigma_\eta^2\right)}{2\beta\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2}.$$

Note, however, that

$$\begin{aligned}\frac{(1-r)^2 (1+\alpha+\beta)^2 \left(1+\sigma_\eta^2\right)}{\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2} &\leq \frac{\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)\left(1+\sigma_\eta^2\right)}{\left(1+\sigma_\eta^2+(1-r)\beta\sigma_\eta^2\right)^2} \\ &\leq \begin{cases} 1 & r \leq \frac{1}{2} \\ \frac{1+\beta}{(1+(1-r)\beta)^2} & r > \frac{1}{2} \text{ and } \beta \leq \frac{2r-1}{1-r} \\ \frac{1}{4r(1-r)} & r > \frac{1}{2} \text{ and } \beta > \frac{2r-1}{1-r} \end{cases},\end{aligned}$$

where the first inequality uses the fact that the left-hand side is increasing in α and that $\alpha \in \left[0, 1/\sigma_\eta^2\right]$, and the second inequality follows from maximizing the left-hand side with respect to σ_η^2, so that if we set

$$\psi \geq \underline{\psi}(r, \beta) = \begin{cases} \frac{1}{2\beta} & r \leq \frac{1}{2} \\ \frac{1+\beta}{2\beta(1+(1-r)\beta)^2} & r > \frac{1}{2} \text{ and } \beta \leq \frac{2r-1}{1-r} \\ \frac{1}{8\beta r(1-r)} & r > \frac{1}{2} \text{ and } \beta > \frac{2r-1}{1-r} \end{cases},$$

then $D(\varphi^\alpha, \alpha_i)(\alpha) \leq 0$ for all $\alpha \in \left[0, 1/\sigma_\eta^2\right]$ and $\sigma_\eta^2 \geq 0$.

***If $\beta = 0$, $D(\varphi^\alpha, \alpha_i)(\alpha) > 0$ always. Prove $\varphi_0 < 0$.

5.2 Proof of Lemma 6:

We obtain the discriminant for the equation $\widetilde{\varphi}$:

$$D\left(\widetilde{\varphi},\alpha\right)\left(\sigma_\eta^2\right)\propto\left(1-r\right)\left(1+\sigma_\eta^2\right)$$
$$-2\left(1+\sigma_\eta^2+\left(1-r\right)\beta\sigma_\eta^2\right)\left(r\left[1+\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2\right]+\left(1-r\right)\beta\right)\psi.$$

Notice that there is a solution for equation (12) only if $\psi\leq\frac{2(1-r)^2}{(1+(1-r)\beta)^2}$ and there is a solution for equation (13) only if $\psi\leq\frac{1-r}{2(r+(1-r)\beta)}\leq\frac{2(1-r)^2}{(1+(1-r)\beta)^2}$. That is why we ought to specify $\psi\leq\frac{1-r}{2(r+(1-r)\beta)}$ for the lemma. Note that $\psi\leq\frac{1-r}{2(r+(1-r)\beta)}$ implies

$$D\left(\widetilde{\varphi},\alpha\right)\left(\sigma_\eta^2,\psi\right)\leq 0\left(\geq 0\right)\qquad\text{when }\sigma_\eta^2\geq\left(\leq\right)\overline{\sigma}_\eta^2.$$

Since

$$D\left(\widetilde{\varphi},\alpha\right)\left(\underline{\sigma}_\eta^2\right)=\frac{16\left(1-r\right)^8\left[1-2r-\beta\left(1+\left(1-r\right)\beta\right)\psi\right]^2}{\left(1+\left(1-r\right)\beta\right)^4\psi^2}\geq 0,$$

it follows that $\underline{\sigma}_\eta^2\leq\overline{\sigma}_\eta^2$ with equality if and only if $1-2r-\beta\left(1+\left(1-r\right)\beta\right)\psi=0$.

5.3 Proof of Proposition 7:

Pick a $\psi>0$ sufficiently small so that

$$\psi<\frac{\left(1-r\right)^2}{\left(1+\left(1-r\right)\beta\right)^2}$$

and in turn $\underline{\sigma}_\eta^2>1$. If $\sigma_\eta^2\geq\underline{\sigma}_\eta^2$, then $\sigma_\eta^2>1$. From equation (5), we obtain $\varphi_1\left(\alpha\right)<0$ and, by equation (6),

$$\varphi_0\left(0\right)=\left[2\left(1-r\right)^2-\left(1+\sigma_\eta^2\right)\left(1+\left(1-r\right)\beta\right)^2\psi\right]\left(1+\beta\right)^2=\widetilde{\varphi}_0\left(1+\beta\right)^2\leq 0,$$

where the last inequality invokes Lemma 6. Thus, from Lemma 3, for $\sigma_\eta^2\geq\underline{\sigma}_\eta^2$, there exists a symmetric Nash equilibrium with $\alpha=0$.

5.4 Proof of Proposition 8:

Pick a $\psi > 0$ sufficiently small so that

$$\psi < \frac{(1-r)^2}{(1+(1-r)\beta)^2}$$

and in turn $\underline{\sigma}_\eta^2 > 1$. Given $\sigma_\eta^2 < \underline{\sigma}_\eta^2$, then

$$\begin{aligned}
\varphi_0(\alpha) &= (1+\beta)^2 \left(1+\sigma_\eta^2\right)\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2 \\
&\times \left\{\frac{2(1-r)^2(1+\alpha+\beta)^2}{(1+\beta)^2\left(1+\sigma_\eta^2\right)\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2}-\psi\right\} \\
&\geq (1+\beta)^2\left(1+\sigma_\eta^2\right)\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2 \\
&\times \left\{\frac{2(1-r)^2}{(1+(1-r)\beta)^2\left(1+\sigma_\eta^2\right)}-\psi\right\} \\
&= (1+\beta)^2\left[1-r\alpha\sigma_\eta^2+(1-r)\left(\alpha+\left(1-r\alpha\sigma_\eta^2\right)\beta\right)\right]^2 \frac{\widetilde{\varphi}_0\left(\sigma_\eta^2\right)}{(1+(1-r)\beta)^2} > 0
\end{aligned}$$

where the first weak inequality follows from the fact that the first term in braces is increasing in α and that $\alpha \geq 0$, and the second strict inequality is given by Lemma 6. But

$$\begin{aligned}
D\left(\varphi^\alpha, \alpha_i\right)(0) &= 4(1-r)^2(1+\beta)^2\left(1+\sigma_\eta^2\right) \\
&\times \left[(1-r)^2(1+\beta)^2\left(1+\sigma_\eta^2\right)-2\beta\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)(1+(1-r)\beta)^2\psi\right] \\
&= 4(1-r)^4(1+\beta)^2\left(1+\sigma_\eta^2\right)\left\{(1+\beta)^2\left(1+\sigma_\eta^2\right)-\frac{4\beta\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)}{1+\underline{\sigma}_\eta^2}\right\} \\
&\geq 4(1-r)^4(1+\beta)^2\left(1+\sigma_\eta^2\right) \\
&\times \min\left[(1+\beta)^2-\frac{4\beta}{1+\underline{\sigma}_\eta^2}, \frac{\left((1+\beta)\underline{\sigma}_\eta^2-(\beta-1)\right)^2}{1+\underline{\sigma}_\eta^2}\right] \\
&\geq 4(1-r)^4(1+\beta)^2\left(1+\sigma_\eta^2\right)\min\left[1+\beta^2, 2\right] \\
&> 0
\end{aligned}$$

where the second equality is obtained by using (12), the first weak inequality follows from the fact that the term in braces is linear in σ_η^2 and that $\sigma_\eta^2 < \underline{\sigma}_\eta^2$, and the second weak inequality follows from minimizing each term in the second square brackets with respect to $\underline{\sigma}_\eta^2$ subject to $\underline{\sigma}_\eta^2 > 1$. Thus, notice that by virtue of Lemma 3, for $\sigma_\eta^2 < \underline{\sigma}_\eta^2$, there does not exist a symmetric Nash equilibrium with $\alpha = 0$.

Recall the fact that if $\sigma_\eta^2 < \underline{\sigma}_\eta^2$, then $\widetilde{\varphi}_0\left(\sigma_\eta^2\right) > 0$. From this it follows that the roots of equation (7) satisfy $\alpha_1(\widetilde{\varphi}) > 0$ and $\alpha_2(\widetilde{\varphi}) < 0$. Therefore, showing $\widehat{\varphi}(\alpha_1(\widetilde{\varphi})) < 0$ is sufficient to show that there exists a symmetric Nash equilibrium with $\alpha_1(\widetilde{\varphi}) > 0$ (this follows because, for $\sigma_\eta^2 < \underline{\sigma}_\eta^2$, $\varphi_0(\alpha) > 0$). Note that $\widehat{\varphi}(\alpha_1(\widetilde{\varphi})) < 0$ if and only if

$$2\left(\widetilde{\varphi}_0 - \widehat{\varphi}_0\right)\left(-\widetilde{\varphi}_2\right) + \left(\widetilde{\varphi}_1 - \widehat{\varphi}_1\right)\left(\widetilde{\varphi}_1 + \sqrt{\widetilde{\varphi}_1^2 - 4\widetilde{\varphi}_2\widetilde{\varphi}_0}\right) > 0$$

which, by formulas (10) and (11), can be rewritten as

$$2\left(1 + \beta\left(\sigma_\eta^2 - 1\right) + \sigma_\eta^2\right)\left(-\widetilde{\varphi}_2\right) + \left(1 + \sigma_\eta^2 + 2\beta\sigma_\eta^2\right)\left(\widetilde{\varphi}_1 + \sqrt{\widetilde{\varphi}_1^2 - 4\widetilde{\varphi}_2\widetilde{\varphi}_0}\right) > 0.$$

The inequality is evidently satisfied when $\beta \leq 1$, or $\beta > 1$ and $\sigma_\eta^2 \geq \frac{\beta-1}{1+\beta}$ because $\widetilde{\varphi}_2 < 0$ and $\widetilde{\varphi}_1 + \sqrt{\widetilde{\varphi}_1^2 - 4\widetilde{\varphi}_2\widetilde{\varphi}_0} > 0$. When $\beta > 1$ and $\sigma_\eta^2 < \frac{\beta-1}{1+\beta}$,

$$\begin{aligned}
&2\left(1 + \beta\left(\sigma_\eta^2 - 1\right) + \sigma_\eta^2\right)\left(-\widetilde{\varphi}_2\right) + \left(1 + \sigma_\eta^2 + 2\beta\sigma_\eta^2\right)\widetilde{\varphi}_1 \\
&\propto 1 + \sigma_\eta^2 + \left(\frac{1 + (1-r)\beta}{(1-r)/r}\sigma_\eta^2 - 1\right)\psi \times \\
&\qquad \left\{\left(1 + (\beta+1)\sigma_\eta^2\right)\left(\left(\frac{1}{r} + \sigma_\eta^2\right)\frac{1 + (1-r)\beta}{(1-r)/r} - 1\right) + \beta\left[1 + (1 + (1-r)\beta)\sigma_\eta^2\right]\right\} \\
&> 1 + \sigma_\eta^2 + \min\left[\left(\frac{1 + (1-r)\beta}{(1-r)/r}\sigma_\eta^2 - 1\right), 0\right] \times \\
&\qquad \frac{\left(1 + \beta\sigma_\eta^2 + \sigma_\eta^2\right)\left(\frac{1 + (1 + (1-r)\beta)\sigma_\eta^2}{(1-r)/r} + \beta\right) + \beta\left[1 + (1 + (1-r)\beta)\sigma_\eta^2\right]}{(1 + (1-r)\beta)^2 / (1-r)^2} \\
&\geq \min\left[1 + \sigma_\eta^2, \frac{1 + \beta^2 + \sigma_\eta^2\left(1 - \beta^2\right)}{(1+\beta)^2}\right] > 0
\end{aligned}$$

where the first strict inequality follows from the fact that the second line is linear in ψ and that $0 < \psi < \frac{(1-r)^2}{(1+(1-r)\beta)^2}$, the weak inequality follows from the fact that the second term in the braces is increasing in r for $\beta > 1$ and that $0 < r < 1$, and the second strict inequality is obtained by the restriction $\sigma_\eta^2 < \frac{\beta-1}{1+\beta}$. In summary, we have shown that if $\alpha_1(\widetilde{\varphi}) > 0$, $\widehat{\varphi}(\alpha_1(\widetilde{\varphi})) < 0$. It also follows that there is a symmetric

Nash equilibrium with $\alpha_1(\widetilde{\varphi}) > 0$, which is unique, for $\sigma_\eta^2 < \underline{\sigma}_\eta^2$.

If, on the other hand, $\sigma_\eta^2 > \overline{\sigma}_\eta^2$, then $\sigma_\eta^2 > \underline{\sigma}_\eta^2$. Thus, from Proposition 7, for $\sigma_\eta^2 > \overline{\sigma}_\eta^2$, there exists a symmetric Nash equilibrium with $\alpha = 0$. Finally, Lemma 6 implies that, when $\sigma_\eta^2 > \overline{\sigma}_\eta^2$, $D(\widetilde{\varphi}, \alpha)\left(\sigma_\eta^2, \psi\right) < 0$ holds, which establishes that there does not exist a symmetric Nash equilibrium with $\alpha > 0$ by invoking Lemma 4. Therefore, there is a unique symmetric Nash equilibrium with $\alpha = 0$ for $\sigma_\eta^2 > \overline{\sigma}_\eta^2$.

5.5 Proof of Proposition 9:

Pick a $\psi > 0$ sufficiently small so that

$$\psi < \min\left[\frac{2(1-r)^2 r}{(1+(1-r)\beta)(1+r(1-r)\beta)}, \frac{1-2r}{(1+(1-r)\beta)\beta}\right].$$

Given $r < \frac{1}{2}$, it is easy to show that

$$\min\left[\frac{2(1-r)^2 r}{(1+(1-r)\beta)(1+r(1-r)\beta)}, \frac{1-2r}{\beta(1+(1-r)\beta)}\right] \leq \frac{(1-r)^2}{(1+(1-r)\beta)^2},$$

and thus $\psi < \frac{(1-r)^2}{(1+(1-r)\beta)^2}$. In that case, it is also easy to verify that

$$\underline{\sigma}_\eta^2 > \max\left[\frac{1-r}{r(1+(1-r)\beta)}, \frac{(1-r)\beta-(1-2r)}{(1-2r)(1+(1-r)\beta)}\right] \geq 1.$$

Thus, from Proposition 7, for $\sigma_\eta^2 \in \left[\underline{\sigma}_\eta^2, \overline{\sigma}_\eta^2\right]$, there exists a symmetric Nash equilibrium with $\alpha = 0$. Recall that, when $\sigma_\eta^2 \geq \overline{\sigma}_\eta^2$, $\widetilde{\varphi}_0\left(\sigma_\eta^2\right) \leq 0$ (by Lemma 6). Note, therefore, that having $\widetilde{\varphi}_1\left(\sigma_\eta^2\right) < 0$ is sufficient to prove the proposition, because then $\widetilde{\varphi}(\alpha) < 0$ for $\alpha > 0$, so that there does not exist a symmetric Nash equilibrium with $\alpha > 0$ (by Lemma 4). Note from equation (9) that $\widetilde{\varphi}_1\left(\sigma_\eta^2\right) < 0$ if and only if

$$\psi < \frac{(1-r)^2\left(\sigma_\eta^2 - 1\right)}{(1+(1-r)\beta)\left(1+\sigma_\eta^2\right)\left[r(1+(1-r)\beta)\sigma_\eta^2 - (1-r)\right]}.$$

But

$$\begin{aligned}
\psi &= \frac{(1-r)\left(1+\overline{\sigma}_\eta^2\right)}{2\left(1+\overline{\sigma}_\eta^2+(1-r)\beta\overline{\sigma}_\eta^2\right)\left(r\left[1+(1+(1-r)\beta)\overline{\sigma}_\eta^2\right]+(1-r)\beta\right)} \\
&\le \frac{(1-r)\left(1+\sigma_\eta^2\right)}{2\left(1+\sigma_\eta^2+(1-r)\beta\sigma_\eta^2\right)\left(r\left[1+(1+(1-r)\beta)\sigma_\eta^2\right]+(1-r)\beta\right)} \\
&< \frac{(1-r)^2\left(\sigma_\eta^2-1\right)}{(1+(1-r)\beta)\left(1+\sigma_\eta^2\right)\left[r(1+(1-r)\beta)\sigma_\eta^2-(1-r)\right]}
\end{aligned}$$

where the equality is merely a rearrangement of equation (13), the weak inequality follows from the fact that the right side of the second line is decreasing in σ_η^2 and that $\sigma_\eta^2 \le \overline{\sigma}_\eta^2$, and the strict inequality uses the fact that the right side of the third line is strictly greater than the right side of the second line for $\sigma_\eta^2 > \max\left[\frac{1-r}{r(1+(1-r)\beta)}, \frac{(1-r)\beta-(1-2r)}{(1-2r)(1+(1-r)\beta)}\right]$ and that $\sigma_\eta^2 \ge \underline{\sigma}_\eta^2$. Hence $\widetilde{\varphi}_1\left(\sigma_\eta^2\right) < 0$ and so there exists a unique symmetric Nash equilibrium with $\alpha = 0$ for $\sigma_\eta^2 \in \left[\underline{\sigma}_\eta^2, \overline{\sigma}_\eta^2\right]$.

5.6 Proof of Proposition 10:

We will prove the proposition under the parameter restriction of $r \ge \frac{1}{2}$ and $\beta > 0$. The proof of the proposition in the case $r > \frac{1}{2}$ and $\beta = 0$ exactly parallels our proof in this case.

First, pick a $\psi > 0$ sufficiently small so that

$$\psi < \min\left[\frac{(1-r)^2}{(1+(1-r)\beta)^2}, \frac{2(1-r)^2 r(1+\beta)}{(1+(1-r)\beta)\left(2+\beta\left(2-r^2\right)+\beta^2(1-r)r\right)}\right].$$

It is easy to show that

$$\underline{\sigma}_\eta^2 > \max\left[1, \frac{2-r+2(1-r)\beta}{r(1+\beta)(1+(1-r)\beta)}\right] > \frac{1-r}{r(1+(1-r)\beta)}.$$

Recall the facts that $\widetilde{\varphi}_2 < 0$, $\widetilde{\varphi}_0\left(\sigma_\eta^2\right) \le 0$ when $\sigma_\eta^2 \ge \underline{\sigma}_\eta^2$, and $D(\widetilde{\varphi}, \alpha)\left(\sigma_\eta^2\right) \ge 0$ when $\sigma_\eta^2 \le \overline{\sigma}_\eta^2$. Note from equation (9) that $\widetilde{\varphi}_1\left(\sigma_\eta^2\right) > 0$ if and only if

$$\psi > \frac{(1-r)^2\left(\sigma_\eta^2-1\right)}{(1+(1-r)\beta)\left(1+\sigma_\eta^2\right)\left[r(1+(1-r)\beta)\sigma_\eta^2-(1-r)\right]}.$$

But

$$\psi = \frac{2(1-r)^2}{(1+(1-r)\beta)^2\left(1+\underline{\sigma}_\eta^2\right)}$$
$$\geq \frac{2(1-r)^2}{(1+(1-r)\beta)^2\left(1+\sigma_\eta^2\right)}$$
$$> \frac{(1-r)^2\left(\sigma_\eta^2-1\right)}{(1+(1-r)\beta)\left(1+\sigma_\eta^2\right)\left[r(1+(1-r)\beta)\sigma_\eta^2-(1-r)\right]}$$

where the equality is merely a rearrangement of equation (12), the first inequality follows from the fact that the right side of the second line is decreasing in σ_η^2 and that $\sigma_\eta^2 \geq \underline{\sigma}_\eta^2$, and the second inequality uses the fact that the right side of the third line is strictly smaller than the right side of the second line for $\sigma_\eta^2 > \frac{1-r}{r(1+(1-r)\beta)}$ and that $\sigma_\eta^2 \geq \underline{\sigma}_\eta^2$. Hence $\widetilde{\varphi}_1\left(\sigma_\eta^2\right) > 0$ and so $\alpha_1(\widetilde{\varphi}) \geq \alpha_2(\widetilde{\varphi}) \geq 0$. We now show that if $\sigma_\eta^2 > 1$, $\alpha_2(\widetilde{\varphi}) \geq 0$ implies $\widehat{\varphi}(\alpha_2(\widetilde{\varphi})) < 0$, while we refer to the proof to Proposition 8 for showing that, in general, $\alpha_1(\widetilde{\varphi}) > 0$ implies $\widehat{\varphi}(\alpha_1(\widetilde{\varphi})) < 0$. Note that $\widehat{\varphi}(\alpha_2(\widetilde{\varphi})) < 0$ if and only if

$$\widetilde{\varphi}_0 - \widehat{\varphi}_0 + (\widetilde{\varphi}_1 - \widehat{\varphi}_1)\,\alpha_2(\widetilde{\varphi}) > 0$$

which, by formulas (10) and (11), can be rewritten as

$$1+\beta\left(\sigma_\eta^2-1\right)+\sigma_\eta^2+\left(1+\sigma_\eta^2+2\beta\sigma_\eta^2\right)\alpha_2(\widetilde{\varphi}) > 0.$$

The inequality is evidently satisfied when $\sigma_\eta^2 > 1$ because $\alpha_2(\widetilde{\varphi}) \geq 0$. Since $\sigma_\eta^2 \geq \underline{\sigma}_\eta^2$ and $\underline{\sigma}_\eta^2 > 1$, it follows that the inequality is satisfied and that $\widehat{\varphi}(\alpha_2(\widetilde{\varphi})) < 0$. It follows directly from Lemma 4 that $\varphi_0(\alpha_i(\widetilde{\varphi})) \geq 0$ is a sufficient condition for the existence of a symmetric Nash equilibrium with $\alpha = \alpha_i(\widetilde{\varphi}) > 0$, where $i = 1, 2$.

Before we proceed, we describe the behavior of $\varphi_0(\alpha)$ as a function of α for ψ sufficiently small and $\sigma_\eta^2 \in \left[\underline{\sigma}_\eta^2, \overline{\sigma}_\eta^2\right]$. Expression (5) implies that $\varphi_0(\alpha)$ can be written

$$\varphi_0(\alpha) = (1+\beta)^2\,\widetilde{\varphi}_0 + \varphi_{01}\alpha + \varphi_{02}\alpha^2$$

where

$$\begin{aligned}\varphi_{02}\left(\sigma_\eta^2\right) &= 2\left(1-r\right)^2 \\ &\quad -\left(1+\beta\right)^2\left(1+\sigma_\eta^2\right)\left[r\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2-\left(1-r\right)\right]^2\psi \\ \varphi_{01}\left(\sigma_\eta^2\right) &= 4\left(1-r\right)^2\left(1+\beta\right) \\ &\quad +2\left(1+\beta\right)^2\left(1+\left(1-r\right)\beta\right)\left(1+\sigma_\eta^2\right)\left[r\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2-\left(1-r\right)\right]\psi\end{aligned} \tag{14}$$

and

$$D\left(\varphi_0,\alpha\right)=8\left(1-r\right)^2r^2\left(1+\beta\right)^2\left(1+\sigma_\eta^2\right)\left[1+\left(1+\beta\right)\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2\right]^2\psi>0.$$

Notice that $\underline{\sigma}_\eta^2>\frac{1-r}{r(1+(1-r)\beta)}$ implies $\varphi_{01}\left(\sigma_\eta^2\right)>0$ for all $\sigma_\eta^2\geq\underline{\sigma}_\eta^2$. Note from equation (14) that $\varphi_{02}\left(\sigma_\eta^2\right)<0$ if and only if

$$\frac{2\left(1-r\right)^2}{\left(1+\beta\right)^2\left(1+\sigma_\eta^2\right)\left[r\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2-\left(1-r\right)\right]^2}<\psi.$$

But

$$\begin{aligned}\psi &= \frac{2\left(1-r\right)^2}{\left(1+\left(1-r\right)\beta\right)^2\left(1+\underline{\sigma}_\eta^2\right)} \\ &> \frac{2\left(1-r\right)^2}{\left(1+\beta\right)^2\left(1+\underline{\sigma}_\eta^2\right)\left[r\left(1+\left(1-r\right)\beta\right)\underline{\sigma}_\eta^2-\left(1-r\right)\right]^2} \\ &\geq \frac{2\left(1-r\right)^2}{\left(1+\beta\right)^2\left(1+\sigma_\eta^2\right)\left[r\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2-\left(1-r\right)\right]^2}\end{aligned}$$

where the equality is merely a rearrangement of equation (12), the first inequality uses the fact that the right side of the second line is strictly smaller than the right side of the first line for $\underline{\sigma}_\eta^2>\frac{2-r+2(1-r)\beta}{r(1+(1-r)\beta)(1+\beta)}$, and the second inequality follows from the fact that the right side of the third line is decreasing in $\sigma_\eta^2>\frac{1-r}{r(1+(1-r)\beta)}$. Hence $\varphi_{02}\left(\sigma_\eta^2\right)<0$.

The proof proceeds by showing that the three inequalities below are satisfied.

a: $-\frac{\widetilde{\varphi}_1}{2\widetilde{\varphi}_2} < -\frac{\varphi_{01}}{2\varphi_{02}} \iff \widetilde{\varphi}_1\varphi_{02} - \widetilde{\varphi}_2\varphi_{01} > 0$. Note that

$$\begin{aligned}\widetilde{\varphi}_1\varphi_{02} - \widetilde{\varphi}_2\varphi_{01} &\propto 2\left(1-r\right)^2\left(1+\sigma_\eta^2+2\beta\sigma_\eta^2\right)\\ &+\left(1+\sigma_\eta^2\right)\left[r\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2-\left(1-r\right)\right]\left(1+\beta\right)\psi\times\\ &\left\{\left(1-r\right)\left(1+\beta\right)+2r/\left(1+\beta\right)+r\left(1+\beta\right)\left(1+\left(1-r\right)\beta\right)\sigma_\eta^4\right.\\ &\left.+\left(1+2r+\left(3-r-r^2\right)\beta+\left(2-r\right)\left(1-r\right)\beta^2\right)\sigma_\eta^2\right\}\\ &>0\end{aligned}$$

where the inequality follows from $\sigma_\eta^2 > \frac{1-r}{r(1+(1-r)\beta)}$.

b: $\varphi_{01}\left(\varphi_{02}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_2\right)+2\varphi_{02}\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)>0$. Note that $\varphi_{01}\left(\varphi_{02}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_2\right)+2\varphi_{02}\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)$ is equal to

$$\begin{aligned}&\kappa\left\{4\left(1-r\right)^4\beta\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)+2\left(1-r\right)^2\left(1+\sigma_\eta^2\right)\lambda_1\left(\sigma_\eta^2\right)\psi\right.\\ &\quad+\left(1+\beta\right)^2\left(1+\left(1-r\right)\beta\right)\left(1+\sigma_\eta^2\right)^2\left[r\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2-\left(1-r\right)\right]^2\psi^2\\ &\quad\times\left[\left(1-r\right)\beta\left(1+\beta\right)+r+r\left(1+\beta\right)^2\left(1+\left(1-r\right)\beta\right)\sigma_\eta^4\right.\\ &\quad\quad\left.\left.+\left(1+\beta\right)\left(2r+\left(1-r^2\right)\beta+\left(1-r\right)^2\beta^2\right)\sigma_\eta^2\right]\right\}\end{aligned}$$

where the factor of proportionality is $4\left(1-r\right)^2$ and

$$\begin{aligned}\lambda_1\left(\sigma_\eta^2\right) &= r\left(2r-1\right)-\left(1-r\right)\left(2+\left(4-3r\right)\beta+2\left(1-r\right)\beta^2\right)\beta\\ &+\left(1+\beta\right)\left(2r\left(3r-1\right)+2\left(1-2r\right)\left(r^2-r-1\right)\beta\right)\sigma_\eta^2\\ &+\left(1+\beta\right)\left(\left(r-1\right)\left(r^2-7r+4\right)\beta^2+\left(1-r\right)^2\left(3r-2\right)\beta^3\right)\sigma_\eta^2\\ &+r\left(1+\beta\right)^2\left(1+\left(1-r\right)\beta\right)\left(6r-1+\beta+2\left(2-r\right)r\beta+\left(2-r\right)\left(1-r\right)\beta^2\right)\left(\sigma_\eta^2\right)^2\\ &+r^2\left(1+\beta\right)^3\left(2+\beta\right)\left(1+\left(1-r\right)\beta\right)^2\left(\sigma_\eta^2\right)^3.\end{aligned}$$

After computing the values of $\lambda_1\left(\sigma_\eta^2\right)$ and $\lambda_1'\left(\sigma_\eta^2\right)$ for $\sigma_\eta^2 = \frac{1-r}{r(1+(1-r)\beta)}$, i.e.,

$$\begin{aligned}\lambda_1\left(\frac{1-r}{r\left(1+\left(1-r\right)\beta\right)}\right) &\propto \frac{\left(1+\left(1-r\right)\beta\left(2+r+\beta\right)\right)}{r}>0\\ \lambda_1'\left(\frac{1-r}{r\left(1+\left(1-r\right)\beta\right)}\right) &\propto \left(1+\beta\right)\left(4+\left(9-4r-3r^2\right)\beta+\left(1-r\right)\left(5+r\right)\beta^2\right)>0\end{aligned}$$

we see that $\sigma_\eta^2 \geq \frac{1-r}{r(1+(1-r)\beta)}$ implies that $\lambda_1\left(\sigma_\eta^2\right)>0$ and in turn $\varphi_{01}\left(\varphi_{02}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_2\right)+2\varphi_{02}\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)>0$.

c: $2\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)\widetilde{\varphi}_2+\left(\varphi_{02}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_2\right)\widetilde{\varphi}_1>0$. Note that

$$2\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)\widetilde{\varphi}_2+\left(\varphi_{02}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_2\right)\widetilde{\varphi}_1=4\left(1-r\right)^2\lambda_2\left(\sigma_\eta^2\right)$$

where

$$\begin{aligned}
\lambda_2\left(\sigma_\eta^2\right) &= 2\left(1-r\right)^4\left(\left(1+\sigma_\eta^2+\beta\sigma_\eta^2\right)^2+\beta^2\left(\sigma_\eta^2\right)^2\right) \\
&-\left(1-r\right)^2\left(1+\sigma_\eta^2\right)\psi\times \\
&\Big\{\left(1-r\right)\left(1+3r+4\left(1-r\right)\beta+\left(1-r\right)\beta^2\right)+r^2\left(1+\beta\right)^2\left(1+\left(1-r\right)\beta\right)^2\left(\sigma_\eta^2\right)^4 \\
&\quad+2\left(1+2r-4r^2+\left(1-r^2\right)\left(4-3r\right)\beta+2\left(2-r\right)\left(1-r\right)^2\beta^2+\left(1-r\right)^3\beta^3\right)\sigma_\eta^2 \\
&\quad-\left(\sigma_\eta^2\right)^2\times\left[6r^2-1-2r+2\left(11r^2-2-2r-5r^3\right)\beta-2\left(1-r\right)\left(3-3r+r^2\right)\beta^3\right. \\
&\qquad\qquad\left.-\left(7-4r-15r^2+16r^3-3r^4\right)\beta^2-\left(1-r\right)^2\left(2-2r+r^2\right)\beta^4\right] \\
&\quad-2r\beta\left(1+\beta\right)\left(1+\left(1-r\right)\beta\right)\left(3r-1-r^2+\left(1-r\right)\left(2r-1\right)\beta\right)\left(\sigma_\eta^2\right)^3\Big\} \\
&+\left(1+\left(1-r\right)\beta\right)\left(1+\sigma_\eta^2\right)^2\left[r\left(1+\left(1-r\right)\beta\right)\sigma_\eta^2-\left(1-r\right)\right]^2\psi^2 \\
&\times\Big\{\left(1-r\right)\beta\left(1+\beta\right)+r+r\left(1+\beta\right)^2\left(1+\left(1-r\right)\beta\right)\left(\sigma_\eta^2\right)^2 \\
&\qquad+\left(1+\beta\right)\left(\left(1-r^2\right)\beta+\left(1-r\right)^2\beta^2+2r\right)\sigma_\eta^2\Big\}.
\end{aligned}$$

For $r\geq\frac{1}{2}$, we now take the limits

$$\lim_{\psi\to0}\lambda_2\left(\underline{\sigma}_\eta^2\right)=\begin{cases}\frac{32(1-r)^{12}r^2(2r-1)(1+\beta)^2}{\lim_{\psi\to0}(1+(1-r)\beta)^8\psi^4} & \text{if } r>\frac{1}{2}\\ \frac{4\beta(1+\beta)^2}{\lim_{\psi\to0}16(2+\beta)^7\psi^3} & \text{if } r=\frac{1}{2}\end{cases}=\infty$$

$$\lim_{\psi\to0}\lambda_2'\left(\underline{\sigma}_\eta^2\right)=\frac{16\left(1-r\right)^{10}r^2\left(12r-5\right)\left(1+\beta\right)^2}{\lim_{\psi\to0}\left(1+\left(1-r\right)\beta\right)^6\psi^3}=\infty$$

$$,\lim_{\psi\to0}\lambda_2''\left(\underline{\sigma}_\eta^2\right)=\frac{160\left(1-r\right)^8r^2\left(3r-1\right)\left(1+\beta\right)^2}{\lim_{\psi\to0}\left(1+\left(1-r\right)\beta\right)^4\psi^2}=\infty,$$

$$\lim_{\psi\to0}\lambda_2'''\left(\underline{\sigma}_\eta^2\right)=\frac{240\left(1-r\right)^6r^2\left(4r-1\right)\left(1+\beta\right)^2}{\lim_{\psi\to0}\left(1+\left(1-r\right)\beta\right)^2\psi}=\infty.$$

$$\lim_{\psi\to0}\lambda_2^{(4)}\left(\underline{\sigma}_\eta^2\right)=240\left(1-r\right)^4r^2\left(6r-1\right)\left(1+\beta\right)^2>0$$

and compute the value of $\lambda_2^{(5)}\left(\sigma_\eta^2\right)$ for $\sigma_\eta^2 = \underline{\sigma}_\eta^2$, i.e.,

$$\begin{aligned}\lambda_2^{(5)}\left(\underline{\sigma}_\eta^2\right) &= 120r^2\left(1+\beta\right)\left(1+\left(1-r\right)\beta\right)^2\psi\times \\ &\quad \left\{\left(1-r\right)^2\left(12r-1\right)\left(1+\beta\right)\right. \\ &\quad \left.-\left(1+\left(1-r\right)\beta\right)\left(2+\left(1+6r-3r^2\right)\beta+\left(1-r\right)\left(5r-1\right)\beta^2\right)\psi\right\}.\end{aligned}$$

It is easy to see that $\lambda_2^{(6)}\left(\sigma_\eta^2\right)>0$ and for $n\geq 7$, $\lambda_2^{(n)}\left(\sigma_\eta^2\right)=0$. Thus, if $r\geq\frac{1}{2}$, there exists a range of ψ values, $(0,\psi\left(r,\beta\right))$, such that $\lambda_2\left(\sigma_\eta^2\right)>0$ for $\sigma_\eta^2\geq\underline{\sigma}_\eta^2$ and, in particular, for $\sigma_\eta^2\in\left[\underline{\sigma}_\eta^2,\overline{\sigma}_\eta^2\right]$.

Second, set $\psi>0$ so that

$$\psi<\min\left[\frac{\left(1-r\right)^2}{\left(1+\left(1-r\right)\beta\right)^2},\frac{2\left(1-r\right)^2r\left(1+\beta\right)}{\left(1+\left(1-r\right)\beta\right)\left(2+\beta\left(2-r^2\right)+\beta^2\left(1-r\right)r\right)},\psi\left(r,\beta\right)\right].$$

That is, we ensure that the three inequalities are satisfied. Note that $\alpha_1\left(\widetilde{\varphi}\right)<\alpha_1\left(\varphi_0\right)$ if and only if

$$\left(\widetilde{\varphi}_1\varphi_{02}-\widetilde{\varphi}_2\varphi_{01}\right)\sqrt{D\left(\varphi_0,\alpha\right)}+\varphi_{01}\left(\varphi_{02}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_2\right)+2\varphi_{02}\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)>0,$$

which is satisfied when the first two inequalities holds, and $\alpha_2\left(\widetilde{\varphi}\right)\geq\alpha_2\left(\varphi_0\right)$ if and only if

$$\left(\widetilde{\varphi}_1\varphi_{02}-\widetilde{\varphi}_2\varphi_{01}\right)\sqrt{\widetilde{\varphi}_1^2-4\widetilde{\varphi}_2\widetilde{\varphi}_0}\leq 2\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)\widetilde{\varphi}_2+\left(\varphi_{02}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_2\right)\widetilde{\varphi}_1,$$

which, after invoking the third inequality, can be rewritten as

$$\left(\varphi_{00}\widetilde{\varphi}_2-\varphi_{02}\widetilde{\varphi}_0\right)^2+\left(\widetilde{\varphi}_1\varphi_{02}-\widetilde{\varphi}_2\varphi_{01}\right)\left(\varphi_{00}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_0\right)\geq 0$$

But

$$\varphi_{00}\widetilde{\varphi}_1-\varphi_{01}\widetilde{\varphi}_0=-2\left(1-r\right)^2\left(1+\beta\right)\left(1+\beta\left(\sigma_\eta^2-1\right)+\sigma_\eta^2\right)\widetilde{\varphi}_0\geq 0, \tag{15}$$

because $\sigma_\eta^2\geq\underline{\sigma}_\eta^2$ and $\underline{\sigma}_\eta^2>1$, and $\widetilde{\varphi}_0\left(\sigma_\eta^2\right)\leq 0$ when $\sigma_\eta^2\geq\underline{\sigma}_\eta^2$. Thus, inequality (15) holds and in turn $\varphi_0\left(\alpha_i\left(\widetilde{\varphi}\right)\right)\geq 0$ for $i=1,2$. Therefore, there exist symmetric Nash equilibria with $\alpha=\alpha_1\left(\widetilde{\varphi}\right)>0$ and $\alpha=\alpha_2\left(\widetilde{\varphi}\right)$. Finally, note that $\alpha_2\left(\widetilde{\varphi}\right)=0$ and $\alpha_2\left(\widetilde{\varphi}\right)=\alpha_1\left(\widetilde{\varphi}\right)>0$.

To complete the proof, note that from Proposition 7, there exists also a symmetric

Nash equilibrium with $\alpha = 0$.

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