A Welfare Criterion with Endogenous Welfare Weights for Belief Disagreement Models*

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Abstract

While belief disagreement models have played an important role in explaining bubbles, over-trading, and speculation, measuring social welfare in those models has been a challenge. This is because the social planner may not know the objective probabilities or whose subjective beliefs to elect under belief disagreements. We propose a novel welfare criterion that endogenously determines sensible welfare weights based on competitive equilibrium allocation as a benchmark. Applying it to several models with heterogeneous beliefs, we show how regulation in moderation, rather than tight control of the market, can be optimal even in the presence of heterogeneous beliefs.

Key words: welfare criterion, heterogeneous beliefs, risk sharing, financial regulation

JEL codes: D61, D62, D84, G12, G14.

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1 Introduction

Recurring financial crises, such as the dot-com bubble and crash at the turn of the millennium and the US housing bubble followed by the subprime mortgage crisis of 2007-8, have offered a sobering lesson that markets can fuel wild risk taking with severe consequences. This is somewhat distressing in that it may reflect not institutional imperfections in risk sharing but speculative motives that run counter to insurance motives (Shiller, 2006; Cheng, Hong and Scheinkman, 2010). Naturally, these events have motivated an extensive reassessment of current principles and practices of financial regulations. For example, Posner and Weyl (2012) even assert that the government needs to regulate financial innovations the way as it regulates new drugs. Such concerns have also convincingly made it imperative for researchers to go beyond positive theories of speculative bubbles, leverage cycles, and other related phenomena. Indeed, there is a quickly growing consensus that for these models to be practically useful, they also ought to have clear and compelling policy implications.

The literature, however, has not provided much guidance about the extent to which regulators should control the financial markets (Whitehouse, 2009). From an academic perspective, this is not too surprising. While heterogeneous beliefs have played an important role in positive theories for market anomalies, including speculation, bubbles, or over-leveraging, it is generally not clear which probability measure is appropriate for evaluating welfare and in turn, the effects of a policy. Therefore, normative implications of such models are difficult to obtain. This paper contributes to the discussion by proposing a novel way of assessing welfare in models with heterogeneous beliefs.

Before presenting our welfare criterion, we set the stage by briefly reviewing the history of welfare economics and general equilibrium theory. In a seminal work, Arrow (1964) extended the welfare theorems to economies with uncertainty. In particular, he established that any Pareto optimal allocation can be decentralized as a competitive equilibrium allocation if the markets for securities are complete and if regularity conditions are satisfied. In particular, his results apply to economies in which agents maximize expected utility under subjective beliefs (Savage, 1954). These results have created a presumption that as long as markets are complete, the strong optimality properties of economies with certainty will carry over to economies under uncertainty. However, other general equilibrium theorists have recognized that Arrow's (1964) generalization has some unappealing features when agents' subjective beliefs differ from one another. A series of articles by von Weizsacker (1962), Dreze (1970), Starr (1973), Harris (1978), and Hammond (1981) have noted that when traders have heterogeneous beliefs, a competitive equilibrium features less-than-perfect risk sharing even though it is Pareto optimal. Intuitively, perfect risk sharing requires individuals to equate their relative marginal utilities across all states. However, in a competitive equilibrium with complete markets, individuals equate their marginal utilities times their probability assessments for each state. In turn, the different beliefs of agents generate speculation, which creates deviations from the perfect risk sharing benchmark (Simsek, 2012).

The lack of perfect risk sharing by itself does not necessarily generate a rationale for revisiting the usual Pareto criterion. However, an independent decision theory literature—e.g., Hylland and Zeckhauser (1979), Mongin (1997, 2016), Gilboa, Samet, and Schmeidler (2004)—identifies conceptual problems with applying the Pareto criterion in these environments, which they have referred to as the "spurious unanimity problem." Intuitively, when there are heterogeneous beliefs, the competitive equilibrium, even in a negative sum game, can be unanimously preferred to the perfect risk sharing allocation for reasons (or beliefs) that are not unanimous.

Notwithstanding these conceptual problems, models with heterogeneous beliefs quickly grew popular in the finance literature due to its success in explaining numerous puzzling phenomena in financial markets, including stock bubbles and excess trading. Of course, traders may speculate if they have different opinions about the value of the assets, and the differences of opinions may be due to different information among traders;[1] hence, it is tempting to attribute speculations to the latter. But the literature on no-trade theorems (e.g., Milgrom and Stokey, 1982; Tirole, 1982; Brunnermeier, 2009) shows that asymmetric information alone cannot produce the high trading volume found in the data. Indeed, if all agents are rational and share common priors, asymmetry in information does not produce any trade. One way to bypass the no-trade result would be to relax the assumption of common priors and work with a model with heterogeneous prior beliefs.

But then, evaluating social welfare in the presence of belief heterogeneity has been a challenge in itself. This is because the social planner may not know the objective probabilities or not know whose subjective beliefs are correct, if any. Not only the decision theory literature we noted earlier, but the finance theory literature on disagreement models of financial markets has also identified conceptual difficulty

[1] An extensive literature in finance studies markets in which agents have differing opinions not because of different information, but because there is bounded rationality or real ambiguity regarding the "true" probability measure. See Xiong (2013) for a literature review.

with making welfare statements in these models (see, for example, Sims, 2009; Che and Sethi, 2014). In response to these issues, Brunnermeier, Simsek, and Xiong (2014, henceforth 'BSX') and Gilboa, Samuelson, and Schmeidler (2014) have recently proposed welfare criteria that could be applied to environments where there are heterogeneous beliefs.

If the planner does not know which beliefs to adopt, it is only sensible to require welfare statements to be robust across all reasonable beliefs. While BSX (2014) and our paper both originate from this same root and both work within the class of linear social welfare functions (SWF), there are important differences. BSX (2014) propose a purely utilitarian SWF that assigns equal weights to all agents. On the contrary, we concede to the fact that welfare analysis is sensitive to particular Pareto weights being used, and try to provide a rationale for the choice of these weights. In particular, we propose to endogenously determine sensible Pareto weights based on competitive equilibrium allocation as a benchmark. One of our main contributions is to show that this (seemingly technical) difference can lead to considerably different welfare and, in turn, policy implications.

To delineate our idea, consider the following example. Amy and Bart disagree about tomorrow's state, whether it will be state A or B. Amy believes that they are more likely to be in state A, whereas Bart believes the opposite. Such belief heterogeneity creates voluntary speculative motives, which in turn leads to betting between Amy and Bart barring a regulation on betting. In the end, only one of them can be correct and whoever was right will have gained at the expense of the other. Applying standard Pareto criterion here is troublesome for there is a "spurious unanimity." On the other hand, the utilitarian SWF will imply that no betting is always superior to allowing betting, regardless of the extent of belief disagreement. This is because, under Amy's beliefs, her utility loss from no betting would be less than Bart's utility gain from it. Perhaps surprisingly, we show that, if belief disagreement is large enough, banning speculation may not warrant higher social welfare when we use sensible weights to define our SWF (Proposition 2).

When the planner uses Amy's beliefs to evaluate welfare, Amy's utility in the (unconstrained) allocation will be higher than Bart's. In this sense, the subjective beliefs provide guidance on who has high reservation utility and accordingly, deserves high Pareto weight in defining the SWF. In this spirit, in our proposed welfare criterion, we use each agent's expected utility under a (subjective) measure as a constraint on his Pareto weight in defining the SWF with respect to the (subjective) beliefs.

Pareto weights chosen in such a way can lead to considerably different welfare implications than the utilitarian SWF, since they often operate as a countervailing force to the welfare dominance of tight control of the market. For example, when the planner evaluates welfare under Amy's beliefs, since Amy's utility in the (unconstrained) allocation will be higher than Bart's, banning speculation will be like a lump-sum transfer from the rich agent (Amy) to the poor agent (Bart). At the same time, Amy will have higher weight than Bart in defining the SWF under Amy's beliefs.

Risk aversion, or equivalently diminishing marginal utility, would imply that Amy's loss from banning speculation will be less than Bart's gain in utility terms. Thus, the utilitarian SWF would say that banning speculation is welfare improving. On the other hand, our SWF under Amy's beliefs would overweigh Amy's loss and underweigh Bart's gain, rendering the welfare benefits of banning speculation ambiguous. Indeed, through a series of applications in this paper, we show that for many practical models with heterogeneous beliefs, our welfare criterion continues to yield welfare implications, consistent with our welfare analysis of Amy and Bart's betting example.

The remainder of this article is organized as follows. In Section 2, we introduce the model and necessary definitions. We review the BSX welfare criterion and describe our welfare criterion. In Section 3, we derive formal welfare implications of our criterion in symmetric or two-state environments. Section 4 applies our welfare criterion to a standard general equilibrium model of bubbles with heterogeneous beliefs and short-sales constraints, in which the planner seeks the optimal degree of trading restrictions. with heterogeneous beliefs and short-sales constraints as government policies. Section 5 concludes. Mathematical derivations not covered in the main text are relegated to the Appendix.

2 A Welfare Criterion with Heterogeneous Beliefs

2.1 The Setting

Consider an endowment economy with I agents denoted by $i \in \mathcal{I} = \{1, \cdots, I\}$, a single consumption good, two periods denoted by $t \in \{1, 2\}$, and S states in period 2 denoted by $s \in \mathcal{S} = \{1, \cdots, S\}$. Agents have stochastic endowments in period 2, and they only consume in period 2. They trade in financial markets in period 1. For the sake of simplicity, suppose that the financial markets are complete in the sense that Arrow securities for each state s are available for trade. Agents are subjective expected utility maximizers with *potentially different beliefs*.[2] Let $\pi_i(s) \geq 0$ denote agent i's subjective probability for state s. We say that agents i and j have heterogeneous beliefs if $\pi_i(s) \neq \pi_j(s)$ for some state $s \in \mathcal{S}$. Let $\mathbf{C} = \{C_i(s)\}_{i,s}$ denote the consumption allocation, where $C_i(s)$ is agent i's consumption in state s.

Then agent i will choose her allocation to maximize $E_i[U_i(C_i)]$ subject to the budget constraint, where the utility function $U_i(\cdot)$ is increasing, continuously differential, and strictly concave. The subscript i following E indicates that the expectation operator E is evaluated under agent i's beliefs. Let $\mathbf{C}^* = \{C_i^*(s)\}_{i,s}$ denote the competitive equilibrium allocation. If agents share homogeneous beliefs—i.e., $\pi_i(s) = \pi(s)$ for all states $s \in \mathcal{S}$—then the competitive equilibrium allocation $\mathbf{C}^*$ must, as is well known, be characterized by the following first-order conditions:

$$\frac{U_i'(C_i(s))}{U_j'(C_j(s))} = \frac{\zeta_i}{\zeta_j} \tag{1}$$

for any i, j, where $i \neq j$ and ζ_i (ζ_j) denote the Lagrange multiplier from agent i's (j's) constrained optimization. Thus, $\mathbf{C}^*$ features full risk sharing in the sense that individuals regard their relative marginal utilities as constant across all states.

However, the competitive equilibrium allocation no longer features full risk sharing in the presence of heterogeneous beliefs. The first-order conditions are simply changed to

$$\frac{\pi_i(s)U_i'(C_i(s))}{\pi_j(s)U_j'(C_j(s))} = \frac{\nu_i}{\nu_j} \tag{2}$$

where ν_i (ν_j) denotes the new Lagrange multiplier for agent i (j). Thus, individuals equate their relative marginal utilities multiplied by their probability assessments so that they are constant across states. In turn, the competitive equilibrium features imperfect risk sharing. Intuitively, heterogeneous beliefs induce trades based on belief disagreements, which create deviations from the full risk sharing benchmark.

2.2 The BSX Welfare Criterion

This paper's criterion is best understood by comparing it with the recent criterion offered by BSX so that we start by describing the BSX criterion.

Ideally, and especially in the presence of heterogeneous beliefs among agents, the planner should use the objective distribution to evaluate welfare. However, this approach runs into serious difficulties in practice because the planner might not know

[2] The sources for different beliefs can be different prior beliefs, different information during the Bayesian updating process, or distortions in the updating (BSX, 2014).

the objective belief. To circumvent this problem, we need a robust welfare criterion. In this spirit, BSX propose a criterion that requires the planner to use *all reasonable beliefs*. According to their definition, the set of reasonable beliefs contains all possible convex combinations of the agents' beliefs. In particular, they say that an allocation $\mathbf{C}$ is *belief-neutral inefficient* if it is inefficient according to any reasonable belief. They also propose two versions of this criterion: welfare-function-based version and Pareto-efficiency-based version. Since this paper offers an alternative welfare-function-based criterion, we focus on the first version of the BSX criterion.

Welfare-function-based version (BSX, p. 1762–1763): Suppose the social planner aggregates individuals' utilities according to a particular SWF,

$$W\left(E_h\left[U_1\left(C_1\right)\right], \ldots, E_h\left[U_I\left(C_I\right)\right]\right), \tag{3}$$

where the subscript h denotes a convex combination of the agents' beliefs such that agent i's belief is assigned weight h_i; i.e., $\pi_h = \sum_i h_i \pi_i$, with $h_i \geq 0$ and $\sum_i h_i = 1$. For example, if the SWF is a weighted sum of agents' expected utilities, then the aggregation becomes

$$W_h\left(\mathbf{C}; \lambda\right) = \sum_{i=1}^{I} \lambda_i E_h\left[U_i\left(C_i\right)\right], \tag{4}$$

where λ_i is the weight assigned to agent i and E_h indicates that expectations are evaluated under a common measure π_h. Under the utilitarian SWF, all agents are assigned equal weights, and the aggregation becomes

$$W_h\left(\mathbf{C}; \lambda\right) = \sum_{i=1}^{I} E_h\left[U_i\left(C_i\right)\right]. \tag{5}$$

Then the allocation $\mathbf{C}$ is belief-neutral dominated by another allocation $\mathbf{C}'$ if and only if it results in a lower social welfare for any reasonable belief π_h. In this manner, the welfare-function-based approach allows for a direct comparison between $\mathbf{C}$ and $\mathbf{C}'$. Although this criterion (like any robust criterion for decision-making) gives an incomplete ranking of social allocations, it yields clear and intuitive welfare predictions in many settings in which speculations driven by heterogeneous beliefs play an important role.

2.3 Our Proposed Welfare Criterion

We offer an alternative criterion that is similar in spirit to the welfare-function-based version of the BSX criterion. However, there are several differences between our criterion and the BSX criterion. First, we propose to use each agent's belief as a reasonable belief, which is only slightly different than using each convex combination.[3] Second and more fundamental, we try to endogenously determine the set of sensible 'Pareto' weights based on some equilibrium allocation that is taken as a benchmark. To the contrary, BSX use identical weights across all agents.

In particular, our welfare criterion is based on the set of social welfare functions W^i, each of which evaluates utility under the common beliefs of individual agent i,

$$W^i\left(\mathbf{C};\lambda^i\right) = \sum_{j=1}^{I} \lambda_j^i E_i\left[U_j\left(C_j\right)\right], \tag{6}$$

where $\lambda^i = \left\{\lambda_j^i\right\}_j$ is the vector of Pareto weights to be used with the SWF W^i and E_i is the expectation operator under a common measure π_i.

We can apply our welfare criterion by using the following procedure:

(1) We start from a benchmark allocation $\mathbf{C}' = \{C_i'(s)\}_{i,s}$. For the purpose of this paper, we let this benchmark allocation be the competitive equilibrium allocation $\mathbf{C}^*$. As described above, this allocation features less-than-perfect risk sharing, which permits an interesting welfare analysis à la our criterion. This allocation is associated with a set of welfare levels $W^i(\mathbf{C}^*;\lambda^i)$ in (6). Note that the individual W^i's are evaluated under the beliefs of agent i, for every $i \in \mathcal{I}$, and the choice of λ^i may depend on measure π^i being used.

We make welfare comparisons by comparing the welfare under alternative allocations using the planner's welfare functions $\{W^i(\mathbf{C};\lambda^i)\}_i$.

(2) We then define a set of Pareto vectors $\mathbf{\Lambda}(\pi_i,\mathbf{C}^*)$ as follows. Let $\lambda^i \in \mathbf{\Lambda}(\pi_i,\mathbf{C}^*)$ if the consumption allocation vector $\mathbf{C}^i$ that maximizes $W^i(\cdot;\lambda)$ satisfies

$$E_i\left[U_j\left(C_j^i\right)\right] \geq E_i\left[U_j\left(C_j^*\right)\right], \qquad \forall j \in \mathcal{I}. \tag{7}$$

Definition 1 λ^i *is said to satisfy the* <u>*subjective Pareto principle*</u> *with respect to* π_i *and* $\mathbf{C}^*$ *(i.e.,* λ^i *belongs to* $\mathbf{\Lambda}(\pi_i,\mathbf{C}^*)$*) if the planner endowed with the weight vector* λ^i *and with the belief* π_i *would choose a welfare-maximizing allocation* $\mathbf{C}^i$ *that would*

[3] This itself does not generate a significant result. For example, if one allocation is belief-neutral dominated by another allocation by one agent's belief, say π_i, it will be dominated by any convex combination belief involving this belief π_i.

make every agent (at least non-strictly) better off under the beliefs of agent i, compared to the benchmark allocation $\mathbf{C}^*$.

The constraints (7) narrow down the set of allocations, whereby perfect risk sharing is featured, that will be considered as welfare-improving.[4] In turn, they serve as constraints on the set of Pareto vectors that will be considered as sensible, since such allocations are associated with maximizing W^i for some choice of λ. Presumably, if the choice of λ is sensible with respect to the beliefs of agent i, maximizing W^i would induce the planner to make every agent (at least non-strictly) better off under the beliefs of agent i.

In this sense, we can say that the set of allocations satisfying (7), or equivalently the set of Pareto vectors $\mathbf{\Lambda}(\pi_i, \mathbf{C}^*)$, satisfy a variant of the standard Pareto principle, in which the planner respect people's tastes, but impose on them common beliefs, that we call the subjective Pareto principle.

(3) We then define the following notion of subjective dominance:

Definition 2 *An allocation* <u>$\mathbf{C}'$ *welfare-improves* $\mathbf{C}^*$ *with respect to measure* π_i</u> *if there exists* $\lambda \in \mathbf{\Lambda}(\pi_i, \mathbf{C}^*)$ *such that*

$$W^i(\mathbf{C}'; \lambda) > W^i(\mathbf{C}^*; \lambda). \tag{8}$$

The role of $\mathbf{\Lambda}(\pi_i, \mathbf{C}^*)$ becomes clear here. For this definition to make sense, λ cannot be arbitrary because there is always some vector of Pareto weights λ that would satisfy (8). Therefore, we appeal to the variant Pareto principle to limit us to a set of weight vectors $\mathbf{\Lambda}(\pi_i, \mathbf{C}^*)$ for which there exist allocations C^i that improve the utility of each individual agent, as defined in (7) under beliefs of agent i.[5]

(4) Finally, we can define the robust (or unconditional) welfare improving criterion:

Definition 3 *An allocation* <u>$\mathbf{C}'$ *welfare-improves* $\mathbf{C}^*$</u> *if it welfare-improves* $\mathbf{C}^*$ *with respect to every agent's measure* $\{\pi_i\}_i$.

This welfare criterion would be the welfare analogue to the concept of complete risk sharing as standard neoclassical welfare criterion is to that of Pareto optimality. It is clear that condition (7) is not useful for defining an unconditional improving criterion,

[4] In what follows, however, $\mathbf{C}^i$, i.e., the welfare-maximizing allocations for some combination of beliefs π_i and weights λ, do not play a direct role anymore.

[5] Note however that $\mathbf{C}'$ itself does not need to satisfy the inequality (7) in place of $\mathbf{C}^i$, i.e., (8) does not necessarily imply (7).

since inequalities (7) can never hold $\forall i, j$ with at least one strict inequality, as long as the benchmark is a competitive allocation (which is standard Pareto efficient). Nevertheless, our contribution is to highlight how these inequalities can offer much guidance on choosing sensible welfare weights to be used when evaluating welfare under a common belief.

Next, we note that there will always be a continuum of Pareto vectors satisfying the subjective Pareto principle with respect to π_i and $\mathbf{C}^*$. This is because it is always feasible to satisfy (7) and also make at least one agent strictly better off under the beliefs of agent i, as long as the benchmark features less-than-perfect risk sharing. In this case, every choice of whose welfare to strictly improve by how much translates into a choice of Pareto vector satisfying the subjective Pareto principle.

To effectively proceed with analytically characterizing the implications of our welfare criterion, we further restrict the collection of weight vectors to satisfy belief-neutrality, which can be defined as follows:

Definition 4 *Let $\{\upsilon_k^i\}_k$ be the distribution of expected utilities under a common measure π_i, and $\{\lambda_k^i\}_k$ be the vector of Pareto weights to be used with the SWF W^i. Then the collection of weight vectors $\{\lambda_k^i\}_k, i \in \mathcal{I}$, satisfy belief-neutrality if there is a function ω such that*

$$\lambda_j^i = \omega\left(\upsilon_j^i; \left\{\upsilon_k^i\right\}_k\right), \qquad \forall i, j \in \mathcal{I}.$$

Hence the planner has a predetermined function that sets the welfare weights when assessing welfare with respect to an individual belief, where the only information needed to form welfare weights is the vector of agents' expected utilities at the benchmark allocation. While the restriction is mild in that generically there will still be a large nonempty subset of Pareto weights to be entertained, belief-neutrality allows us to derive coherent and explicit welfare implications of heterogeneous beliefs for many canonical models.

Our criterion provides an alternative instrument of assessing welfare in models with heterogeneous beliefs as several subsequent examples will show. Moreover, we came up with this approach from following underlying insight. Suppose that we are currently comparing the allocations according to agent i's beliefs, i.e., $h = i$. Then, the subjective Pareto principle requires the weights tilting towards agent i's belief. Hence, when we are using agent i's belief, there is some benefit from redistributing wealth towards agent i. However, when we are using another agent j's beliefs, for the same reason there is also some benefit from redistributing wealth towards agent j. The social planner should not favor any individual in an *ad hoc* manner. Hence, the only way to honor each agent's belief and comprise is to allow some speculation. Indeed, our criterion allows the agents to speculate to some degree rather than an extreme version of policy intervention such as market shutdown.

2.4 An Example: The Bet Between Joe and Bob

We now apply our criterion to analyze the example of the bet between Joe and Bob from Kreps (2012, p. 193) as BSX did. Providing a simple analysis has several benefits. From the comparison of the two criteria, we expect that readers can see better how the two criteria differ and complement one another and how to implement our criterion. Furthermore, we show that for risk-neutral agents the weights are endogenously made equal to the BSX's utilitarian welfare function. Directly quoting BSX's text (p.1754), the Joe-Bob example reads as:

> Consider a bet between two famous economic theorists, Joe and Bob. One day, Joe and Bob argued over the contents of a pillow. Joe maintained that the pillow had a natural-fiber filling, and Bob thought a polyester filling was more likely. Joe assessed with probability 0.9 that the pillow had natural down and Bob assessed the probability at 0.1. They decided to construct a bet as follows: If the pillow had natural down, Bob would pay Joe \$100, but if it had artificial down, Joe would pay Bob \$100. They could discover the truth only by cutting the pillow open, which would destroy it.
>
> They agreed that the winner would replace the pillow at a cost of \$50. Clearly both Joe and Bob preferred the bet relative to no betting at all, as each expected to make a net profit of \$35 after deducting the cost of replacing the pillow. This bet was desirable from each individual's perspective, and thus it Pareto dominated no betting under the standard Pareto principle. However, the outcome of the bet was worrisome—it led to a wealth transfer between Joe and Bob and a perfectly good pillow being destroyed.

Suppose that both Joe and Bob are risk neutral, $u_{Joe}(w) = w$ and $u_{Bob}(w) = w$, and the social planner uses the utilitarian social welfare function for a reasonable belief. Then, the welfare function will be given as

$$W(E^h[u_{Joe}],\ E^h[u_{Bob}]) = E^h[w_{Joe} + w_{Bob}] = w_{Joe} + w_{Bob}.$$

Hence, it is obvious that without any betting, and regardless of the probability measure the social planner adopts, the social welfare is simply the sum of Joe's and Bob's initial wealth. The bet causes a transfer of \$100 between them and the pillow's destruction. The money transfer has no effect on the social welfare regardless of its direction or the probability measure the social planner adopts to evaluate the welfare. However, replacing the pillow incurs a sure cost of \$50 and, therefore, makes the bet a negative-sum game for every reasonable, common probability measure used to evaluate Joe's and Bob's expected utilities. Thus, the status quo allocation is belief-neutral superior to the bet.

Our criterion differs from that of BSX, but it also determines in a different manner that the betting is inferior to no betting. The analysis takes two steps. First it asks, "Can we propose an allocation that improves on the benchmark allocation for both Joe and Bob?" Second, "What weights are feasible to the social planner to make the Pareto improvement under the proposed allocation?" When this two-step procedure is applied, the social planner uses either Joe's belief or Bob's belief.

The benchmark allocation is that Joe and Bob agree to bet and the pillow is destroyed. Showing this is simple. With betting, there are two states: the pillow had natural down 'state N' or a polyester filling 'state P.' If the state is N, Joe enjoys 50 utils but Bob suffers -100 utils; in the state P, the opposite is the case. Joe thinks $\Pr(N) = .9$ but Bob thinks $\Pr(P) = .9$: i.e., $\pi_J(N) = 0.9$ and $\pi_B(P) = 0.9$. Hence, Joe expects

$$E^J[u_J] = .9 \times (100 - 50) + .1 \times (-100) = 35$$

and this is the same for Bob based on Bob's own belief. Under no betting, both receive zero. Consequently, they will voluntarily agree to bet.

Suppose that the social planner adopted Joe's belief for the welfare evaluation. Then the constraint is the Bob's utility evaluated at Joe's belief. This is $.9 \times (-100) + .1 \times (100 - 50) = -85$. This means that from the perspective of the social planner (who adopted Joe's belief) Bob should have not bet. So, the issue is 'Can we propose an allocation T to Joe and $-T$ to Bob so that the total resource is zero but both parties agree to no betting?' In this sense the two constraints to the planner are given by

$$T \geq 35 = E^J[u_J] = E^J[w_J]$$
$$-T \geq -85 = E^J[u_B] = E^J[w_B]$$

which requires $35 \leq T \leq 85$. And, the set of weights requires that

$$\lambda_J T + \lambda_B(-T) = -T + 2\lambda_J T \geq 0 \Longleftrightarrow \lambda_J \geq \frac{1}{2}.$$

If we carry out a similar analysis using Bob's belief, we will derive $\lambda_B \geq \frac{1}{2}$. Hence, we derive the equal weight on both, which justifies the utilitarian welfare function that the BSX explicitly assumed. Moreover, using the utilitarian welfare function, the social planner will find it better to force no betting because the net welfare for any $T \in [35, 85]$ will be below zero.

3 Application Part I: Two-state symmetric models

Now let us apply our welfare criterion to a series of simple models with heterogeneous beliefs.

3.1 Example I. A symmetric economy with two-agents

As a first example, let us consider a simplest economy with heterogeneous beliefs. Let (U, π, Y) represent an agent type with utility function U, belief distribution π, and stochastic endowment Y. There are two types of agents, 1 and 2. The economy is symmetric in that there are equal numbers of the two types of agents having the same utility function, i.e., $U_i(C) = U(C)$. Assume that for each state $s \in \mathcal{S}$ there exists a different state $s' \in \mathcal{S}$ such that $\pi_i(s') = \pi_j(s)$ and $Y_i(s') = Y_j(s)$ for $i \in \mathcal{I} = \{1, 2\}$, $j \neq i$. Then, there will be three sets that partitions the state space $\mathcal{S}$:

$$\begin{aligned}
\mathcal{S}^o &= \{s \in \mathcal{S} \colon\ \pi_1(s)/\pi_2(s) = 1\}, \\
\mathcal{S}^1 &= \{s \in \mathcal{S} \colon\ \pi_1(s)/\pi_2(s) > 1\}, \\
\mathcal{S}^2 &= \{s \in \mathcal{S} \colon\ \pi_1(s)/\pi_2(s) < 1\}.
\end{aligned}$$

The symmetry in the economy allows a one-to-one function $\psi(s_1)$ mapping from states $s_1 \in \mathcal{S}^1$onto $\mathcal{S}^2$ such that

$$\pi_i(\psi(s_1)) = \pi_j(s_1), \quad Y_i(\psi(s_1)) = Y_j(s_1).$$

First note that the competitive equilibrium requires

$$\pi_1(s)U'[C_1^*(s)] = \pi_2(s)U'[C_2^*(s)]$$

and the symmetry asks for $C_i^*(\psi(s_1)) = C_j^*(s_1)$ for $s_1 \in \mathcal{S}^1$. Using this symmetry, we find that the social planner has the following welfare measure for both $i, j = 1, 2$ and $j \neq i$

$$W^i = \lambda^i E_i[U(C_i)] + (1 - \lambda^i) E_i[U(C_j)]$$

where λ^i satisfies the robustness principle. And, it is easy to show that the benchmark competitive equilibrium allocation is not sustained as a social optimum because there is a feasible alternative allocation $\mathbf{C}^{**}$ that welfare improves $\mathbf{C}^*$.

Proposition 1 *Consider the symmetric economy with the two types of agents introduced in this subsection. Then, there is a feasible allocation* $\mathbf{C}^{**}$ *that welfare improves the competitive equilibrium* $\mathbf{C}^*$ *and that has the following properties:*

(i) $\mathbf{C}^{**}$ *satisfies the symmetry on consumption plan;*

(ii) $\mathbf{C}^{**}$ *imposes more risk sharing than* $\mathbf{C}^*$ *but not perfect risk sharing: i.e.,*

$$0 < \left| \frac{U'[C_1^{**}(s)]}{U'[C_2^{**}(s)]} - 1 \right| < \left| \frac{U'[C_1^{*}(s)]}{U'[C_2^{*}(s)]} - 1 \right| \quad \textit{for } \forall s \notin \mathcal{S}^o$$

(iii) $\mathbf{C}^{**}$ *satisfies the 'Pareto' property: If* $W^i(\mathbf{C}^{**}) < W^i(\mathbf{C})$, *then* $W^j(\mathbf{C}^{**}) > W^j(\mathbf{C})$, $j \neq i$ *for all feasible allocations* $\mathbf{C}$.

The intuition behind the above result is as follows. Note that the welfare weight assigned to each agent depends on his reservation utility in the benchmark allocation. This implies that, when using a particular agent's subjective beliefs for evaluating welfare, he himself will command the bigger weight. Essentially, belief disagreement leads to disparity in which welfare weights are deemed sensible. Since the planner wants robustness to whose beliefs are correct, it can be welfare optimal to give some way to what agents want, i.e., speculation, and require more, not perfect, risk sharing than the competitive equilibrium.

3.2 Example II. Two-state case

Here we consider a simple situation with only two-states $\mathcal{S} = \{1, 2\}$ and two types of agents. We further simplify this example with $\sum_i \pi_i(s) = 1$ for each state: e.g. Agent 1 believes that state 1 will occur with 30% probability and state 2 with 70%, and Agent 2 believes those probabilities are 70% and 30%, respectively. Because the state distribution is binary, we can reduce the belief representation to a single parameter $\pi(1)$. Without loss of generality, subjective beliefs are ordered such that $\pi_1(1) \geq \pi_2(1)$.

We assume that $U_i(C) = U(C)$ and perfect risk sharing in the endowment structure with $Y_i(s) = Y/2$ and thus the aggregate endowment is constant regardless of the state, i.e., $\sum_s Y_i(s) = 1$. Then, in competitive equilibrium we can characterize the benchmark allocation as

$$C_i^*(1) = U'^{-1}[\mu_i P(1)/\pi_i(1)] \quad \text{and} \quad C_i^*(2) = U'^{-1}[\mu_i P(2)/\pi_i(2)]$$

where $P(s)$ is the state s-contingent price of the good and μ_i is the Lagrangian multiplier applied to agent i. The feasibility constraint requires $C_1(s) + C_2(s) \leq Y_1(s) + Y_2(s) = Y$, which implies $P(1) = P(2)$. Then, we can derive explicitly that

$$C_i^*(i) = \frac{Y}{2} + \Delta \quad \text{and} \quad C_i^*(j) = \frac{Y}{2} - \Delta \quad \text{for } \Delta \in [-Y/2,\ Y/2]$$

which means that the competitive equilibrium will induce a betting between the agents with heterogeneous beliefs.

See it to that no feasible allocation robustly dominates the competitive equilibrium. Put differently, using each agent's subjective belief, there is no way to make both agents better off without harming at least one compared to the competitive equilibrium. Can we have a welfare-improving government intervention? If so, when? We find that the market shutdown, the seemingly better solution all the time, is effective in welfare-improving the competitive equilibrium only when differences in opinion are not large.

To see this point, let us work in more details for the above two-state case. First, note that the symmetry built in the two-state case yields the following sets of weights for the social planner:

$$\Lambda(\pi_1, \mathbf{C}^*) = \left\{(\lambda, 1-\lambda) : \lambda \in [\underline{\lambda}, 1-\overline{\lambda}]\right\}$$
$$\Lambda(\pi_2, \mathbf{C}^*) = \left\{(1-\lambda, \lambda) : \lambda \in [\underline{\lambda}, 1-\overline{\lambda}]\right\}$$

with $\underline{\lambda} > 1/2$ and $\overline{\lambda} < 1$. In words, if the social planner uses an agent i's belief for the welfare assessment, the set of weights must be given such that either agent receives a sufficiently high weight when her belief is used (which means a sufficiently small weight to the agent whose belief is not used) in order to have any welfare improvement over the competitive equilibrium. Without loss of generality, we choose the welfare weights $\lambda^1 = (\underline{\lambda}, 1-\underline{\lambda})$ and $\lambda^2 = (1-\underline{\lambda}, \underline{\lambda})$ to evaluate welfare with respect to the measures π_1 and π_2, respectively. We see that the welfare at the competitive equilibrium with respect to each measure is

$$W^i(\mathbf{C}^*; \lambda^i) = [\underline{\lambda}\pi + (1-\underline{\lambda})(1-\pi)]\, U(Y/2+\Delta) + [(1-\underline{\lambda})\pi + \underline{\lambda}(1-\pi)]\, U(Y/2-\Delta)$$

for $i = 1, 2$ and the welfare under no-trade policy is $U(Y/2)$. Using the Taylor series expansion around $Y/2$, the condition when the market shutdown is better than allowing the bet can be expressed as

$$U(Y/2) > U(Y/2) + (2\underline{\lambda} - 1)(2\pi - 1)U'(Y/2)\Delta + U''(Y/2)\Delta^2$$

which is equivalent to

$$-\frac{U''(Y/2)}{U'(Y/2)} > \frac{(2\underline{\lambda}-1)(2\pi-1)}{\Delta}. \tag{9}$$

Therefore, for π close enough to $1/2$, the claimed inequality will hold. Quintessentially, no-trade policy can welfare improve the competitive equilibrium only if the subjective beliefs are not much heterogeneous.

Proposition 2 *There exists a positive number $\xi < 1$ such that no trade policy welfare improves the competitive equilibrium only if $|\pi_1(s) - \pi_2(s)| < \xi$.*[6]

Intuitively, large differences of opinion create speculative motives and thus shutting down the market can improve the welfare with respect to one belief because the agent holding the other belief can then be stopped from undertaking the "bad" trade. But, note that there is a trade-off here. No trade would mean a utility loss for the "correct" agent but a gain for the "incorrect" agent. An increase in the heterogeneous beliefs implies a bigger weight on the correct agent so that the main force at work is the welfare loss associated with no-trade policy. As a result, no-trade policy cannot welfare improve the competitive equilibrium for large enough differences of opinion.

Now let us move on to identify the alternative allocation $\mathbf{C}^{**}$ that welfare improves the competitive equilibrium. Formally, we are looking for an allocation satisfying

$$W^i(\mathbf{C}^{**}, \lambda^i) > W^i(Y, \lambda^i) \quad \text{for } \lambda^i \in \Lambda\left(\pi^i, \mathbf{C}^*\right)$$

for $i = 1, 2$. Let $\Delta(s) = |C_1(s) - Y_1(s)| = |C_2(s) - Y_2(s)|$ denote the trading volume in the claims to consumption in state s. In implementing the optimal allocation $\mathbf{C}^{**}$,

[6] For a concrete example, when we think of $U(C) = \log(C)$, then we can compute the threshold $\xi = 0.892$, which allows a generous range in the opinion spectrum for the welfare improvement.

we restrict asset trades to $\Delta^{**} = |C_1^{**} - Y/2| = |C_2^{**} - Y/2|$ to impose the optimal risk-sharing arrangement. The insight to implement the optimal allocation $\mathbf{C}^{**}$ by restricting the trade volume, instead of shutting down the market completely, should be applied well to more general settings beyond the symmetric two-state case. That said, we wrap up this application by the following proposition.

Proposition 3 *Consider the symmetric economy with two agents and two states where the endowment imposes perfect risk sharing. Then, we can find a welfare-improving alternative allocation which optimally bear a risk-sharing arrangement by restricting the asset trades to*

$$\Delta(s) \leq \Delta^{**}(s) = |C_1^{**}(s) - Y(s)| = |C_2^{**}(s) - Y(s)|$$

where $\mathbf{C}^{**}$ *is the allocation identified in Proposition 1.*

3.3 Example III. A model for monetary policy

As an extension of Example II, here let us apply the welfare criterion to the simple monetary policy model of Sims (2009) which features no money illusion, frictionless markets, no short-sales constraints, and a non-stochastic production function.[7] Here agents have different beliefs about inflation. One important implication of the model is that the differences of opinion about a monetary policy could lead to bubble-like phenomena, specifically overinvesment in the real asset even for a frictionless economy. Evaluating welfare in the model faces difficulty because of the belief heterogeneity. However, applying our welfare criterion, we can identify that betting is detrimental to welfare and thus calls for a regulation. The model can be briefly described as follows.

The economy has two types of agents $i \in \{a, b\}$ who live two periods $t = 1, 2$, and two states of the world $s \in \{f, m\}$. In the f state, the tax backing for bonds is low, and hence inflation is high. In the m state taxes are high and prices are lower. Each agent begin life with an endowment of nominal B_0, and during the first period each agent has an endowment Y units of goods. Each agent has several choices in the first period: They can consume in the first period, C_1. They can finance their consumption and their investment S from their endowment or by selling some of their bonds. Agents can also purchase more bonds.

[7] Note that the goal of this application is to make a relevant point as an exercise for the proposed welfare criterion, neither the full analysis of the model nor any assessment of the model itself. In fact, Sims admitted that the model is extremely simple to make a point, which is useful for our purpose.

Then, the agent i's problem can be written as

$$\max_{C_{i1},B_i,S_i,C_{i2f},C_{i2m}} U(C_1) + \beta\left[\pi_i U(C_{i2f}) + (1-\pi_i)U(C_{i2m})\right] \tag{10}$$

subject to

$$C_{i1} + S_i + \frac{B_i - B_0}{P_1} = Y \tag{11}$$

$$C_{i2s} = \rho S_i + \frac{RB_i}{P_{2s}} - \tau_s + \delta, \quad s = f, m \tag{12}$$

where ρ represents the rental price of capital in the second period; τ_j is the lump-sum tax rate in stage j; δ is the profit dividend; β is the typical discount factor; P_1 and P_{2s} are the price levels at $t = 1, 2$ and state s; R denotes the gross nominal interest rate; and finally, $\pi_i = \Pr[s = f]$ and thus $1 - \pi_i = \Pr[s = m]$.

For an explicit solution, Sims (2009) assume specific functional forms: the CRRA utility function $U(C) = C^{1-\alpha}/(1-\alpha)$ with $\alpha < 1$ (with $U(C) = \ln C$ as a limiting case as $\alpha \to 1$) and a concave production function $g(S)$ which captures diminishing returns from total capital S. The model also assumes that the government fixes R and τ_j, which implies the second period budget constraints are $RB_0/P_{2s} = 2\tau_s$. There is no government action such as taxing, spending, or debt sales in the first period, which gives the market clearing condition of $2B_0 = B_a + B_b$.

Using the first-order condition with respect to S (investment) and B (bond sales), we can analytically derive the competitive equilibrium denoted by a vector of $(C^*_{i1}, C^*_{i2f}, C^*_{i2m}, S^*)$. We use $\pi_a = \pi$ for brief notation and consider the symmetric case, i.e., $\pi_a + \pi_b = 1$. Then, we can derive as follows:

$$\begin{aligned}
C^*_{a1} &= C^*_{b1} = Y - \frac{S^*}{2} \\
C^*_{a2f} &= C^*_{b2m} = \frac{\pi^{1/\alpha}}{\pi^{1/\alpha} + (1-\pi)^{1/\alpha}} g(S^*) \\
C^*_{b2f} &= C^*_{a2m} = \frac{(1-\pi)^{1/\alpha}}{\pi^{1/\alpha} + (1-\pi)^{1/\alpha}} g(S^*) \\
f(\pi) &= \frac{U'(Y - S^*/2)g(S^*)^\alpha}{\beta g'(S^*)}
\end{aligned}$$

where $f(\pi) = \left[\pi^{1/\alpha} + (1-\pi)^{1/\alpha}\right]^\alpha$ and $\pi < 1/2$.

Then, as we did before, our goal is to show that there is an alternative feasible allocation $(C^{**}_{i1}, C^{**}_{i2f}, C^{**}_{i2m}, S^{**})$ that welfare improves the benchmark allocation. While relegating the derivations to the Appendix as a proof for the following proposition, we again find it is so.

Proposition 4 *Consider the simple monetary policy model of Sims (2009) as described above. Then, we find that*

(i) there exists a feasible allocation-investment pair $(C_{i1}^{**}, C_{i2f}^{**}, C_{i2m}^{**}, S^{**})$ *that welfare improves the competitive equilibrium* $(C_{i1}^{*}, C_{i2f}^{*}, C_{i2m}^{*}, S^{*})$*;*

(ii) this alternative allocation imposes a more risk sharing than the competitive equilibrium, but not perfect risk sharing;

(iii) the investment level is lower than the competitive equilibrium, i.e., $S^{**} < S^{*}$*.*

Interesting and important, the welfare optimal plan does not call for perfect risk-sharing but more risk-sharing than the competitive equilibrium. We also find that the competitive equilibrium causes an excess investment, which is socially undesirable, in the presence of differences of opinion about real interest rates. These findings also imply that there is not enough inter-temporal consumption smoothing with the agents thinking that they can buy cheaply consumption goods in the state that they think is more likely to arise.

The optimal policy is similarly characterized as in Proposition 3. The key is to restrict the volume in nominal bond trades to such an extent that implements the welfare optimal allocation-investment pair. For this, let $\Delta(s) = |B_a - B_0| = |B_b - B_0|$ denote the bond trading volume. Then, it can be proved that there exists a welfare improving $\Delta(s) \leq \Delta^{**}(s) = |B_a^* - B_0| = |B_b^* - B_0|$. Therefore, in the model of Sims (2009) we assert that regulating the bond market to a certain extent, neither shutting it down nor leaving it unregulated, renders a better societal outcome.

4 Application Part II: Speculative bubbles and government interventions

In this second part of applications, we pursue two goals, one substantive and the other technical. The substantive goal is to show once again that regulating the financial market modestly can yield a higher social welfare compared to leaving speculative bubbles unregulated when the economy has heterogeneous beliefs. The technical theme is to illustrate how our welfare assessment can be extended to the continuum-player economy. The continuum-player case requires a welfare weight function rather than a vector. We suggest a weighting function that is continuous and consistent with the subjective Pareto principle. In addition, we also compute the welfare function using the BSX's criterion in order to provide further insight about how our criteria and BSX's criteria are different but complement to each other.

Heterogeneous beliefs are known to be a prominent mechanism to produce speculative bubbles together with short-sales constraints (Miller, 1977).[8] Intuitively, this is because the equilibrium prices would reflect the views of the optimistic investors. But the extensive literature on financial bubbles leaves open the question of whether or not asset bubbles are socially bad, if so when it is so, and gives little guidance about to what extent regulators should control it. In this section, let us apply our welfare criterion to a model of bubbles with heterogeneous beliefs and short-sales constraints (Lintner, 1969).

4.1 The Setup

Consider an economy that lasts for two periods. There is one risky asset $Z \sim \mathcal{N}(m, \sigma^2)$ that pays off at the end of the second period. Assume that each agent i initially own one unit of the asset Z and have the same utility function of constant absolute risk aversion (CARA), i.e. $U(C) = -\exp(-\gamma C)$ where γ measures the degree of risk aversion. Agents differ only in that they have different beliefs about Z. Agent i believes that $Z \sim \mathcal{N}(m_i, \sigma^2)$, i.e., they disagree about the mean value of Z. We assume that m_i is uniformly distributed around m over the domain $[m - \kappa, m + \kappa]$. The supply of the asset is constant and equal to Q. Let T_i denote the initial wealth of agent i and p denote the price of the asset. Then, we must have $T_i = p \times 1$ and the budget constraint $p = pH_i + b_i$ where H_i is the number of shares of the asset held by agent i and b_i is the dollar amount in bond (numeraire) agent i chooses. Each agent i's final wealth is given by

$$W_i = b_i R + H_i Z = p\left(1 - H_i\right) R + H_i Z$$

where $R (= 1 + r)$ is the gross rate of return with interest rate r. For simplicity, assuming $R = 1$, the expected return and variance are derived as

$$\widehat{E}\left[W_i\right] = p + H_i\left(\widehat{\mu} - p\right); \qquad \widehat{Var}\left[W_i\right] = H_i^2 \sigma^2.$$

[8]Daniel, Hirshleifer and Subrahmanyam (1998) and Scheinkman and Xiong (2003) derive heterogeneous beliefs from overconfidence, a well documented behavioral bias, and highlight that it can lead to excessive trading and asset price bubbles. More recently, Jung, Kim, Matejka and Sims (2015) study portfolio choice problem with rational inattention. They prove that rational inattention predicts persistent differences of opinion, as investors are constrained on their use of information, and, in turn, discretely distributed portfolio choice.

Hence, the agent's expected utility can be expressed as follows:

$$\begin{aligned}\widehat{E}\left[U\left(W_i\right)\right] &= -\exp\left(-\gamma\widehat{E}\left(W_i\right)+\frac{\gamma^2}{2}\widehat{Var}\left(W_i\right)\right)\\ &= -\exp\left(-\gamma\left[p+H_i\left(\widehat{\mu}-p\right)\right]+\frac{\gamma^2}{2}H_i^2\sigma^2\right).\end{aligned}$$

Each agent will choose H_i to maximize her expected utility:

$$\max_{H_i} -\exp\left(-\gamma\left[p+H_i\left(m_i-p\right)\right]+\frac{\gamma^2}{2}H_i^2\sigma^2\right)$$

subject to $H_i \geq 0$. Solving the (unconstrained) model, the demand function for the asset is derived as

$$H_i^* = \frac{\max\left(m_i-p,0\right)}{\gamma\sigma^2}.$$

4.2 Unregulated Equilibrium

The demand for the risky asset Z equals to its supply in equilibrium, which is given by

$$\int_{\max(p,m-\kappa)}^{m+\kappa} \frac{m_i-p}{\gamma\sigma^2}\frac{dm_i}{2\kappa} = Q$$

and the equilibrium price p^* is derived as

$$p^*\left(\kappa;m,\gamma\sigma^2Q\right) = \begin{cases} p_H^* \equiv m+\kappa-2\sqrt{\kappa\gamma\sigma^2Q} & for\, \frac{\kappa}{\gamma\sigma^2Q}>1 \\ p_L^* \equiv m-\gamma\sigma^2Q & for\, \frac{\kappa}{\gamma\sigma^2Q}\leq 1. \end{cases}$$

We use this unregulated equilibrium as a benchmark for the analysis of impacts of policy interventions.

4.3 Regulated Equilibrium

Now we consider government policies that restrict asset trades. Specifically, the policy ϕ restricts the amount of trading to $H_i - Q \leq \phi$, thereby allowing us policies that prevent asset bubbles. For example, policy $\phi = 0$ shuts down the asset market and thus no trade equilibrium is enforced. The other extreme is policy $\phi = \infty$, which has no restriction except the short-sales constraint. The revised demand function with policy parameter ϕ is as follows:

$$H_i(\phi) = \min\left\{\max\left\{\frac{m_i-p(\phi)}{\gamma\sigma^2},0\right\},\ Q+\phi\right\} \tag{13}$$

where $p(\phi)$ is the equilibrium price under the policy ϕ. Aforementioned, the analysis proceeds for the two distinct cases depending on the benchmark scenarios: Case I. $\frac{\kappa}{\gamma\sigma^2 Q} > 1$ and Case II. $\frac{\kappa}{\gamma\sigma^2 Q} \leq 1$.

■ **Case I:** $\kappa/\gamma\sigma^2 Q > 1$

The first case is that the unregulated equilibrium generate the asset bubble such as $p^* = p_H^* := m + \kappa - 2\sqrt{\kappa\gamma\sigma^2 Q}$. While relegating the detailed mathematical derivations to the Appendix, here let us provide the key result with intuitive explanation.

Lemma 1 *If $\kappa/\gamma\sigma^2 Q > 1$, then the equilibrium price under policy ϕ is derived as*

$$p(\phi) = \begin{cases} p_H^* & \phi \geq \overline{\phi}_1 \\ m + \kappa - \frac{\gamma\sigma^2}{2}(Q+\phi) - 2\kappa\frac{Q}{Q+\phi} & \phi \in \left(\underline{\phi}, \overline{\phi}_1\right) \\ m - \kappa - \gamma\sigma^2(Q+\phi) + 2\sqrt{\gamma\sigma^2\kappa\phi} & \phi \leq \underline{\phi} \end{cases}$$

where the thresholds are defined as

$$\overline{\phi}_1 = \left(2\sqrt{\frac{\kappa}{\gamma\sigma^2 Q}} - 1\right)Q, \quad \underline{\phi} = \left(\sqrt{\frac{\kappa}{\gamma\sigma^2}} - \sqrt{\frac{\kappa}{\gamma\sigma^2} - Q}\right)^2.$$

Intuitively, the equilibrium price will be characterized by the unregulated overpricing equilibrium p_H^* if the regulation is sufficiently loose for $\phi \geq \overline{\phi}_1$. But, once the policy gets tight, the price started to be restricted. Specifically, for an intermediate range of $\phi \in (\underline{\phi}$, the price equilibrium is determined as an interior solution, but for more strict policy of $\phi \leq \underline{\phi}$ it follows a corner solution.

■ **Case II:** $\kappa/\gamma\sigma^2 Q \leq 1$

We analyze the second case that the unregulated equilibrium is equal to $p^* = p_L^* := m - \gamma\sigma^2 Q$, which shows that the price is independent of the dispersion parameter κ. Again we reserve the involved analysis in the Appendix, we provide the result in Lemma 2:

Lemma 2 *If $\kappa/\gamma\sigma^2 Q \leq 1$, then*

$$p(\phi) = \begin{cases} p_L^* & \phi \geq \overline{\phi}_2 \\ m - \kappa - \gamma\sigma^2(Q+\phi) + 2\sqrt{\gamma\sigma^2\kappa\phi} & \phi < \overline{\phi}_2 \end{cases}.$$

where $\overline{\phi}_2 := m - (\gamma\sigma^2 + 1)Q$.

Here the sufficiently strong policy intervention ($\phi < \overline{\phi}_2$) restricts the equilibrium price further down; otherwise, the equilibrium price is the same as the unregulated one.

4.4 Solving for Planner's Problem for Fixed Welfare Weights

By now we have solved for equilibrium price under all possible cases, which makes us move on to study welfare weights. Suppose the planner entertains agent j's beliefs for the purposes of welfare evaluation. For welfare weights $\{\lambda_i\}$, he solves

$$\max \int_{m-\kappa}^{m+\kappa} \lambda_i E_j \left[U \left(W_i \right) \right] \frac{dm_i}{2\kappa}$$

subject to

$$\int_{m-\kappa}^{m+\kappa} W_i \left(Z \right) \frac{dm_i}{2\kappa} = QZ$$

Without loss of generality, let

$$\int_{m-\kappa}^{m+\kappa} \lambda_i \frac{dm_i}{2\kappa} = 1.$$

The first-order conditions imply

$$\frac{\exp \left(\gamma W_i \left(Z \right) \right)}{\exp \left(\gamma W_m \left(Z \right) \right)} = \frac{\lambda_i}{\lambda_m},$$

i.e.,

$$W_i \left(Z \right) = W_m \left(Z \right) + \log \left(\frac{\lambda_i}{\lambda_m} \right)^{1/\gamma}$$

Aggregating $W_i \left(Z \right)$ across all agents and equating it to QZ imply

$$W_m \left(Z \right) + \int_{m-\kappa}^{m+\kappa} \log \left(\frac{\lambda_i}{\lambda_m} \right)^{1/\gamma} \frac{dm_i}{2\kappa} = QZ.$$

Therefore,

$$W_i \left(Z \right) = - \int_{m-\kappa}^{m+\kappa} \log \left(\frac{\lambda_i}{\lambda_m} \right)^{1/\gamma} \frac{dm_i}{2\kappa} + QZ + \log \left(\frac{\lambda_l}{\lambda_m} \right)^{1/\gamma}.$$

Lemma 3 *For welfare weights $\{\lambda_i\}$, the social planner would award agent i*

$$W_i \left(Z \right) = a_i + QZ,$$

where

$$a_i = -\int_{m-\kappa}^{m+\kappa} \log\left(\frac{\lambda_i}{\lambda_m}\right)^{1/\gamma} \frac{dm_i}{2\kappa} + \log\left(\frac{\lambda_i}{\lambda_m}\right)^{1/\gamma}.$$

Lemma 3 implies that agent i will expect from the planner's allocation that

$$\begin{aligned} E_j(W_i) &= a_i + Qm_j \\ Var_j(W_i) &= Q^2\sigma^2 \end{aligned}$$

which in turn leads to

$$\begin{aligned} E_j[U(W_i)] &= -\exp\left(-\gamma E_j(W_i) + \frac{\gamma^2}{2} Var_j(W_i)\right) \\ &= -\exp\left(-\gamma(a_i + Qm_j) + \frac{\gamma^2}{2} Q^2\sigma^2\right). \end{aligned}$$

4.5 Solving for Welfare Weights

We will search for welfare weights such that planner will award agent i expected utilities evaluated with agent j's beliefs proportional to agent i's benchmark utilities evaluated with agent j's beliefs. Hence, we have

$$-\exp\left(-\gamma(a_i + Qm_j) + \frac{\gamma^2}{2} Q^2\sigma^2\right) = \varpi E_j[U_i^*]$$

where $E_j(U_i^*)$ is agent i's benchmark utilities evaluated with agent j's beliefs and ϖ is the proportion parameter. Solving for a_i,

$$a_i = -Qm_j + \frac{\gamma}{2} Q^2\sigma^2 - \frac{1}{\gamma}\log(-\varpi E_j[U_i^*])$$

Similarly,

$$a_m = -Qm_j + \frac{\gamma}{2} Q^2\sigma^2 - \frac{1}{\gamma}\log(-\varpi E_j[U_m^*])$$

Therefore, the difference between agent i's choice and the average's is given by

$$a_i - a_m = \log\left(\frac{E_j[U_m^*]}{E_j[U_i^*]}\right)^{1/\gamma}$$

Recall from the analysis in subsection 4.3 that

$$a_i - a_m = \log\left(\frac{\lambda_i}{\lambda_m}\right)^{1/\gamma}$$

We conclude that the weight is reciprocal to the agent i's benchmark utilities evaluated with j's belief, i.e.,

$$\lambda_i \propto \frac{1}{E_j\left[U_i^*\right]}$$

Intuitively, the social planner would give a lower weight to an agent whose benchmark utility is higher. For $j = i$, agent i finds the highest benchmark utility, which makes the planner gives the lowest weight to the agent i because agent i does impose the least restriction in the social welfare optimization problem. The last step of the analysis is to compute the SWF using the derived weights $\{\lambda_i\}$.

4.6 A Numerical Illustration and Welfare Criteria Comparison

A welfare analysis for this model does not always yield an analytical solution. However, in this subsection we show how we can execute numerically the welfare analysis using our method. For this purpose, let us consider the above model with the following parameter values: $m = 0, \gamma = 1, Q = 1$, $\sigma^2 = 1$, and $\kappa = 2$, and subjective beliefs in this economy about the mean value of Z are uniformly distributed over $[-2, 2]$. Note that we have $\kappa > \gamma\sigma^2 Q$ in this economy so that the belief heterogeneity is large enough to render the short-sales constraints binding in the unregulated equilibrium for the more pessimistic agents. To illustrate how social welfare W^i depends on the common beliefs of individual agent i, we plot our welfare measures as functions of the policy parameter ϕ in Figure 1, with respect to the beliefs of the following nine agents:

$$m_i = -2, \ -1.5, \ -1, \ -0.5, \ 0, \ 0.5, \ 1, \ 1.5, \text{ and } 2.$$

The policy parameter ϕ ranges from 0 (the no-trade policy) to 1.83 (a policy lax enough that the economy is effectively in the unregulated equilibrium beyond this value). In addition, $m_i = -2$ ($m_i = +2$) represents the most pessimistic (optimistic) belief in this economy, while $m_i = 0$ represents the belief of the moderate agent.

[Place Figure 1 about here.]

We first observe that the shape of the social welfare functions differs conspicuously depending on the common beliefs under which we evaluate expected utilities. This is fundamentally due to endogenous variations in welfare weights implied by different beliefs. To see this, we plot in Figure 2 the BSX welfare measures as functions of the policy parameter ϕ for the same nine agents. Recall that the BSX welfare function exogenously assigns Pareto weights, independent of beliefs under which it is being evaluated. In particular, they operationalize their welfare function by assuming equal weights across all agents.

[Place Figure 2 about here.]

A comparison of the two plots reveals clear divergences between the policy rankings based on our welfare criterion and those based on the BSX welfare criterion. The comparison also shows that divergences are most acute when welfares are evaluated under extreme beliefs. Indeed, as we tighten regulation (i.e., as we lower the policy parameter ϕ), our welfare functions evaluated under extreme beliefs (i.e., the first two and the last two panels) trend downward. This trend is non-monotonic—for example, welfare measures initially increase as we move away from the regulated equilibrium—but it is clearly present. On the other hand, as we tighten regulation, BSX welfare functions, regardless of the beliefs under which they are evaluated, trend upward.

From the more pessimistic agent's perspective, the unregulated equilibrium leads to asset overpricing. In the unregulated equilibrium, these agents believe they are benefiting from this arbitrage opportunity at the expense of the more optimistic agents, who are paying too much for the asset. Regulation tames asset overpricing, which, in turn, suppresses the arbitrage opportunity. Thus, the more optimistic agents believe that the more optimistic agents, effectively forced from irrationally paying too much, benefit from tighter regulation at the expense of themselves. Performing an analogous welfare analysis from the more optimistic agent's perspective, it is easy to conclude that, quintessentially, regulations benefit the initial losers at the expense of the initial winners from the perspectives of agents who have extreme beliefs.[9]

As is well known from standard economics, concave utility functions render the benefits to initial losers larger in magnitude than the costs to initial winners. Since BSX welfare functions equally weigh these benefits against these costs, the BSX criterion concludes that regulation is welfare-improving when evaluated under extreme beliefs. In contast, our welfare functions weigh the initial winners more highly than the initial losers, and consequently the weighted benefits do not turn out to justify the weighted costs. We conclude that regulation is usually not welfare-improving when it is evaluated under extreme beliefs.

Next, we observe that our welfare criterion and the BSX welfare criterion agree on the policy ranking when welfares are evaluated under moderate beliefs (i.e., panels

[9]Since Miller (1977), the investor disagreement and short-sales constraints framework has generated a large and successful literature on asset overpricing. In our opinion, it would be interesting to apply our framework to dynamic models of bubbles, such as those proposed by Harrison and Kreps (1978), Scheinkman and Xiong (2003), Hong and Sraer (2011).

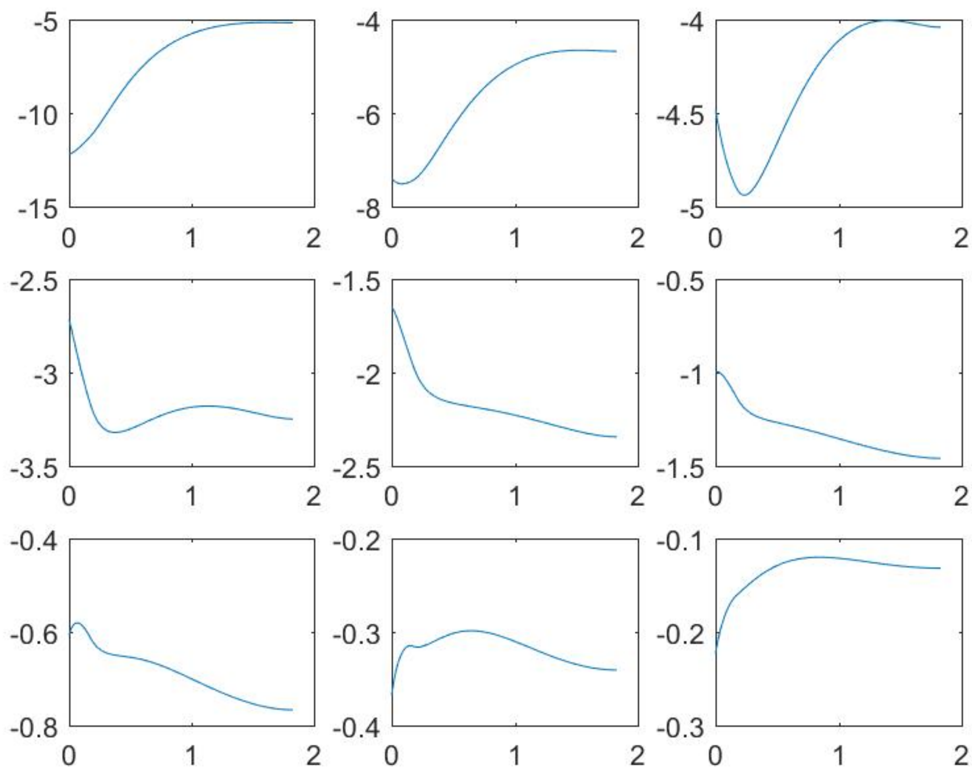
in the middle row). This is so because from the moderate agent's perspective, both optimists and pessimists are losers in the unregulated equilibrium. In addition, our welfare criterion under moderate beliefs does not down-weigh optimists (pessimists) as much as it would under pessimistic (optimistic) beliefs, which allows the benefits from regulation to drive our policy ranking. For the same reason, the BSX welfare criterion under moderate beliefs indicates that regulation is welfare-improving.

Finally, we address the question if our welfare criterion and BSX welfare criterion are significantly different enough so as to propose different optimal policies. Indeed, it is easy to see that the BSX welfare criterion implies that the no–trade policy is robustly welfare-maximizing in the sense that it maximizes welfare under all agents' beliefs. In contrast, our welfare criterion implies that the policy parameter values $\phi \in (1.49, 1.61)$ are robustly welfare-maximizing. Recall that the unregulated equilibrium corresponds to the policy parameter value $\phi = 1.83$. Allowing beliefs to endogenously determine the appropriate Pareto weights, our welfare criterion calls for much more relaxed regulation. But then it also highlights the clear benefits of policy intervention.

5 Concluding Remarks

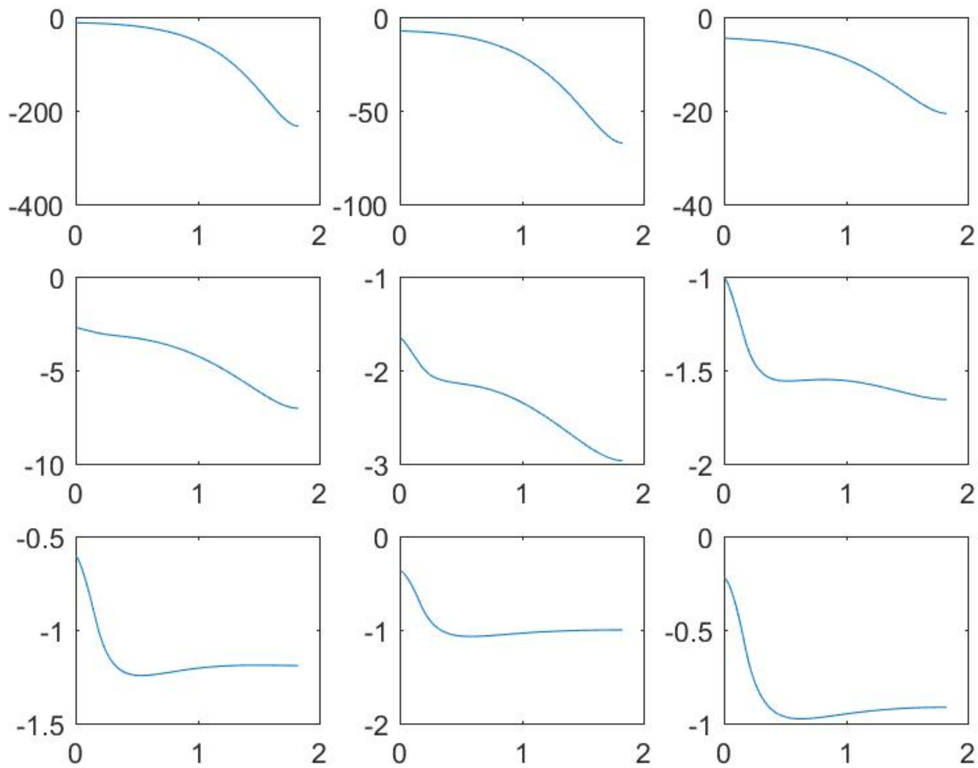
Our welfare criterion builds on the fact that equilibrium outcomes with heterogeneous beliefs inherit the Pareto property, but not complete risk sharing from the case with homogeneous beliefs. We have shown that diverse beliefs, be they due to bounded rationality or real uncertainty over the objective probabilities, engender speculative motives, which impede risk sharing and encourage betting that relies on others in the economy being wrong. In turn, we have argued for the essential nature of regulating markets in the presence of heterogeneous beliefs. The obvious standard neoclassical welfare criterion would say that we do not discriminate between beliefs and preferences and we ought to evaluate everyone's utility with respect to his own probabilities. This paper, along with recent developments in the literature, serves as a point of departure from this perspective and formalizes the real possibility of welfare destruction in unregulated markets. In particular, for prominent economic models with heterogeneous beliefs, welfare analysis applying our welfare criterion points to consistent and precise policy recommendations for regulation in moderation. We hope that it will promote welfare analysis under belief disagreement, which has matured into a key concept in behavioral finance and economics.

Figure 1: Social Welfare Assessment Using Our Welfare Criterion



Note: For these graphs we use following parameters: $m = 0$, $\kappa = 2$, $\gamma = 1$, $Q = 1$, $\sigma^2 = 1$. The top left panel corresponds to the agent with the most pessimistic belief ($m = -2$); the bottom right, the most optimistic agent ($m = +2$); the center, the modest ($m = 0$). The horizontal axis is for the policy parameter ϕ and the vertical axis is the social welfare with agent i's belief.

Figure 2: Social Welfare Assessment Using the Welfare Criterion in BSX (2014)



Note: For these graphs we use following parameters: $m = 0$, $\kappa = 2$, $\gamma = 1$, $Q = 1$, $\sigma^2 = 1$. The top left panel corresponds to the agent with the most pessimistic belief ($m = -2$); the bottom right, the most optimistic agent ($m = +2$); the center, the modest ($m = 0$). The weights are equally applied across all agents following BSX (2014).

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The Appendix: Mathematical Proofs

Proof of Proposition 1.

If $s \in \mathcal{S}^o$ and thus the agents have homogeneous beliefs $\pi_1(s) = \pi_2(s)$, we have $C_1^{**}(s) = C_2^{**}(s) = \frac{Y_1(s)+Y_2(s)}{2}$ and the competitive equilibrium is also characterized by $C_1^{*}(s) = C_2^{*}(s) = \frac{Y_1(s)+Y_2(s)}{2}$.

If $s \notin \mathcal{S}^o$ and thus under heterogeneous beliefs, we obtain $0 < \left|\frac{U'[C_1^{**}(s)]}{U'[C_2^{**}(s)]} - 1\right|$. To see this, note that the social planer maximizes the weighted sum of the two utility functions, $\lambda U^1(C) + (1-\lambda)U^2(C)$, subject to the feasibility condition $\sum_s C(s) \leq \sum_s Y(s)$. Let C^{**} be the solution to this optimization problem. Then, the first-order conditions imply that

$$\frac{U'[C_1(s)]}{U'[C_2(s)]} = \frac{\pi_1(s)(1-\lambda) + \pi_2(s)\lambda}{\pi_1(s)\lambda + \pi_2(s)(1-\lambda)}$$

and $C_i^{**}(\psi(s_1)) = C_j^{**}(s_1)$ for all $i \neq j$ and $s_1 \in \mathcal{S}^1$. For $\pi_1(s) \neq \pi_2(s)$, we have the claimed result.

And, if the social planner's optimal allocation deviates from the competitive allocation ($C^{**} \neq C^*$), we will be able to say that $\left|\frac{U'[C_1^{**}(s)]}{U'[C_2^{**}(s)]} - 1\right| < \left|\frac{U'[C_1^{*}(s)]}{U'[C_2^{*}(s)]} - 1\right|$ for any positive λ because of

$$\frac{\pi_1(s)(1-\lambda) + \pi_2(s)\lambda}{\pi_1(s)\lambda + \pi_2(s)(1-\lambda)} < \frac{\pi_1(s)}{\pi_2(s)}.$$

Next, let us establish the symmetry of consumption plan. For this purpose, we expand $W^1(C)$ and show it ends up with becoming $W^2(C)$:

$$
\begin{aligned}
&W^1(C) = \lambda \sum_s \pi_1(s)U[C_1(s)] + (1-\lambda) \sum_s \pi_1(s)U[C_2(s)] \\
&= \lambda \left[\sum_{s \in \mathcal{S}^1} \pi_1(\psi(s))U[C_1(\psi(s))] + \sum_{s \in \mathcal{S}^2} \pi_1(\psi^{-1}(s))U[C_1(\psi^{-1}(s))] + \sum_{s \in \mathcal{S}^o} \pi_1(\psi(s))U\left[(Y_1(s)+Y_2(s))/2\right] \right] \\
&+(1-\lambda) \left[\sum_{s \in \mathcal{S}^1} \pi_1(\psi(s))U[C_2(\psi(s))] + \sum_{s \in \mathcal{S}^2} \pi_1(\psi^{-1}(s))U[C_2(\psi^{-1}(s))] + \sum_{s \in \mathcal{S}^o} \pi_1(\psi(s))U\left[(Y_1(s)+Y_2(s))/2\right] \right] \\
&= \lambda \left[\sum_{s \in \mathcal{S}^2} \pi_2(s)U[C_2(s)] + \sum_{s \in \mathcal{S}^1} \pi_2(s)U[C_2(s)] + \sum_{s \in \mathcal{S}^o} \pi_2(s)U\left[(Y_1(s)+Y_2(s))/2\right] \right] \\
&+ (1-\lambda) \left[\sum_{s \in \mathcal{S}^1} \pi_2(s)U[C_1(s)] + \sum_{s \in \mathcal{S}^2} \pi_2(s)U[C_1(s)] + \sum_{s \in \mathcal{S}^o} \pi_2(s)U\left[(Y_1(s)+Y_2(s))/2\right] \right] \\
&= \lambda \sum_s \pi_2(s)U[C_2(s)] + (1-\lambda) \sum_s \pi_2(s)U[C_1(s)] = W^2(C)
\end{aligned}
$$

Hence, the symmetry on consumption allocations establishes $W^1(C^*) = W^2(C^*)$ and $W^1(C^{**}) = W^2(C^{**})$.

The subjective Pareto principle in this symmetric model implies that the sets of Pareto weights are given by

$$
\begin{aligned}
\Lambda(\pi_1, \mathbf{C}^*) &= \left\{(\lambda, 1-\lambda) : \lambda \in [\underline{\lambda}, 1-\overline{\lambda}]\right\} \\
\Lambda(\pi_2, \mathbf{C}^*) &= \left\{(1-\lambda, \lambda) : \lambda \in [\underline{\lambda}, 1-\overline{\lambda}]\right\}
\end{aligned}
$$

with $\underline{\lambda} > 1/2$ and $\overline{\lambda} < 1/2$. By definition, this requires that

$$
\begin{aligned}
W^1(C) &= \lambda E_1[U(C_1)] + (1-\lambda)E_1[U(C_2)] > \lambda E_1[U(C_1^*)] + (1-\lambda)E_1[U(C_2^*)] \\
W^2(C) &= \lambda E_2[U(C_2)] + (1-\lambda)E_2[U(C_1)] > \lambda E_2[U(C_2^*)] + (1-\lambda)E_2[U(C_1^*)]
\end{aligned}
$$

which implies $W^1(C) + W^2(C) > W^1(C^*) + W^2(C^*)$ because $W^1(C) + W^2(C) = 2W^i(C) > 2W^i(C^*) = W^1(C^*) + W^2(C^*)$.

Lastly, to show that $W^i(C^{**}) < W^i(C^*)$ implies $W^j(C^{**}) > W^j(C^*)$ for $j \neq i$, suppose in negation that $W^j(C^{**}) \leq W^j(C^*)$. Then, we have a contradiction to the derivation that $W^1(C) + W^2(C) > W^1(C^*) + W^2(C^*)$. □

Proof of Proposition 2.

As we delivered the sketch of the proof in the text, here we describe more details for complete proof. Taylor series expansion around $Y/2$ are approximated as

$$U(Y/2+\Delta) \approx U(Y/2) + \Delta U'(Y/2) + \Delta^2 U'(Y/2)$$
$$U(Y/2-\Delta) \approx U(Y/2) - \Delta U'(Y/2) + \Delta^2 U'(Y/2)$$

And we found that the desired claim can be expressed as

$$-\frac{U''(Y/2)}{U'(Y/2)} > \frac{(2\underline{\lambda}-1)(2\pi-1)}{\Delta}$$

which is the case for π close enough to $1/2$. To show that this fact, we take the limit of the right-hand-side of the above inequality and then apply l'hospital's rule as follows:

$$\lim_{\pi\to1/2} \frac{(2\underline{\lambda}-1)(2\pi-1)}{\Delta} = 2\lim_{\pi\to1/2} \frac{\frac{d\underline{\lambda}}{d\pi}(2\pi-1)+(2\underline{\lambda}-1)}{dC_i^*(i)/d\pi}$$

(recalling $C_i^*(i) = \frac{Y}{2}+\Delta$, we have $\frac{d\Delta}{d\pi} = \frac{dC_i^*(i)}{d\pi}$). If $\pi = 1/2$, then $C_i^*(i) = Y/2$. According to the budget constraint, $d\mu/d\pi = 0$ by implicit differentiation. Thus,

$$\lim_{\pi\to1/2} \frac{dC_i^*(i)}{d\pi} = \lim_{\pi\to1/2} \frac{1}{U''\left[C_i^*(i)\right]}\left(\frac{d\mu}{d\pi} - U'\left[C_i^*(i)\right]\right)\frac{1}{\pi} = -2\frac{U'(Y/2)}{U''(Y/2)}.$$

Note that

$$\underline{\lambda} = \frac{U'(Y-\Psi)}{U'(\Psi)+U'(Y-\Psi)}$$

where Ψ represents the competitive equilibrium allocation under the homogeneous beliefs and thus $\Psi = U^{-1}(\pi U\left[C_i^*(i)\right] + (1-\pi)U\left[C_i^*(j)\right])$. Then, $\lim_{\pi\to1/2}(2\underline{\lambda}-1) = 0$. And,

$$\lim_{\pi\to1/2}\frac{d\underline{\lambda}}{d\pi} = -\lim_{\pi\to1/2}\frac{U''(Y-\Psi)U'(\Psi)+U'(Y-\Psi)U''(\Psi)}{\left[U'(\Psi)+U'(Y-\Psi)\right]^2}\times$$
$$\frac{1}{U'(\Psi)}\left\{U\left[C_i^*(i)\right] - U\left[C_i^*(j)\right] + \pi\frac{dU\left[C_i^*(i)\right]}{d\pi} + (1-\pi)\frac{dU\left[C_i^*(j)\right]}{d\pi}\right\} = 0$$

Hence, $\lim_{\pi\to1/2}\frac{(2\underline{\lambda}-1)(2\pi-1)}{\Delta} = 0$ and π close enough to $1/2$ implies no trade welfare improves the competitive equilibrium. $\square$

Proof of Proposition 3.

We can construct an argument that is analogous to the proof of Proposition 1 using the symmetry. The risk-sharing arrangement by restricting asset trades to $\Delta(s) \leq \Delta^{**}(s) = |C_1^{**}(s) - Y(s)| = |C_2^{**}(s) - Y(s)|$ welfare improves the competitive equilibrium. As the essence of the proof is already described in the proof of Proposition 1, here we leave the details out. □

Proof of Proposition 4.

Again, the spirit of the proof is the same as for Proposition 1. The set of the robust weights are given by

$$\begin{aligned}
\Lambda(\pi_a, \mathbf{C}^*) &= \left\{(\lambda, 1-\lambda) : \lambda \in [\underline{\lambda}, 1-\overline{\lambda}]\right\} \\
\Lambda(\pi_b, \mathbf{C}^*) &= \left\{(1-\lambda, \lambda) : \lambda \in [\underline{\lambda}, 1-\overline{\lambda}]\right\}
\end{aligned}$$

with $\underline{\lambda} > 1/2$ and $\overline{\lambda} < 1/2$. The social planner maximizes the sum of the welfare functions, $W^a(C) + W^b(C)$ subject to

$$\begin{aligned}
C_{a1} + C_{b1} + S &= 2Y \\
C_{a2s} + C_{b2s} &= g(S) \quad \text{for } s = f, m
\end{aligned}$$

Let (C^{**}, S^{**}) be the optimal solution to this problem. Then, the first-order conditions imply that

$$\begin{aligned}
C_{a1}^{**} = C_{b1}^{**} &= Y - \frac{S^{**}}{2} \\
C_{a2f}^{**} = C_{b2m}^{**} &= \frac{[\lambda\pi + (1-\lambda)(1-\pi)]^{1/\alpha}}{[\lambda\pi + (1-\lambda)(1-\pi)]^{1/\alpha} + [\lambda(1-\pi) + (1-\lambda)\pi]^{1/\alpha}} g(S^{**}) \\
C_{a2m}^{**} = C_{b2f}^{**} &= \frac{[\lambda(1-\pi) + (1-\lambda)\pi]^{1/\alpha}}{[\lambda\pi + (1-\lambda)(1-\pi)]^{1/\alpha} + [\lambda(1-\pi) + (1-\lambda)\pi]^{1/\alpha}} g(S^{**}) \\
f(\lambda\pi + (1-\lambda)(1-\pi)) &= \frac{U'(Y - S^{**}/2) g(S^{**})^\alpha}{\beta g'(S^{**})}
\end{aligned}$$

Note that $\frac{U'(Y-S^{**}/2)g(S^{**})^\alpha}{\beta g'(S^{**})}$ is strictly increasing in S. Thus, $S^{**} < S^*$ can only be true if and only if

$$f(\lambda\pi + (1-\lambda)(1-\pi)) < f(\pi),$$

which is guaranteed by $f'(\pi) < 0$ and $\pi < \lambda\pi + (1-\lambda)(1-\pi) < 1/2$. □

Proof of Lemma 1.

From (13), we can see that if

$$\frac{m+\kappa-p}{\gamma\sigma^2} \leq Q+\phi$$

then $p = p_H^*$. Observe that this is true if

$$\phi \geq \left(2\sqrt{\frac{\kappa}{\gamma\sigma^2 Q}}-1\right) Q := \overline{\phi}_1.$$

Hence, if $\kappa/\gamma\sigma^2 Q > 1$ and $\phi \geq \overline{\phi}_1$, then $p = p_H^*$.

If $\phi < \overline{\phi}_1$, and if $p > m-\kappa$, the demand for the risky asset Z equals to its supply, which is given by

$$\int_p^{p+\gamma\sigma^2(Q+\phi)} \frac{m_i-p}{\gamma\sigma^2} dm_i + \int_{p+\gamma\sigma^2(Q+\phi)}^{m+\kappa} (Q+\phi)\, dm_i = 2\kappa Q$$

and the equilibrium price $p(\phi)$ is derived as

$$p(\phi) = m+\kappa - \frac{\gamma\sigma^2}{2}(Q+\phi) - 2\kappa\frac{Q}{Q+\phi}.$$

Observe that $p > m-\kappa$ if

$$2\kappa\phi - \frac{\gamma\sigma^2}{2}(Q+\phi)^2 > 0.$$

Define

$$\begin{aligned} f(\phi) &:= 2\kappa\phi - \frac{\gamma\sigma^2}{2}(Q+\phi)^2 \\ &= \left(2\kappa - \gamma\sigma^2 Q\right)\phi - \frac{\gamma\sigma^2}{2}Q^2 - \frac{\gamma\sigma^2}{2}\phi^2. \end{aligned}$$

Observe that $f(0) < 0$, $f'' < 0$ and $f'(0) = 2k - \gamma\sigma^2 Q > 0$ (by assumption, $\kappa/\gamma\sigma^2 Q > 1$). Define, further, $\underline{\phi}$ as the smaller solution of $f(\phi)$, i.e.,

$$\underline{\phi} := \frac{(2\kappa - \gamma\sigma^2 Q) - 2\sqrt{\kappa(\kappa - \gamma\sigma^2 Q)}}{\gamma\sigma^2} = \left(\sqrt{\frac{\kappa}{\gamma\sigma^2}} - \sqrt{\frac{\kappa}{\gamma\sigma^2} - Q}\right)^2 > 0$$

where the inequality is true by assumption, $\kappa/\gamma\sigma^2 Q > 1$. Define also ϕ^{UB} as the larger solution of $f(\phi)$.

Therefore, we need $\phi \in \left(\underline{\phi}, \phi^{UB}\right)$ so that what we have assumed, i.e., $p > m - \kappa$, is true, and in turn, we still need to check if $\overline{\phi}_1 < \phi^{UB}$. Compute

$$\begin{aligned} f'\left(\overline{\phi}_1\right) &= 2\kappa - \gamma\sigma^2 Q - \gamma\sigma^2\overline{\phi}_1 \\ &= 2\sqrt{\kappa\gamma\sigma^2 Q}\left(\sqrt{\frac{\kappa}{\gamma\sigma^2 Q}} - 1\right) > 0 \end{aligned}$$

where the inequality is true again by assumption, $\kappa/\gamma\sigma^2 Q > 1$. This shows that indeed $p > m - \kappa$ as long as $\phi \in \left(\underline{\phi}, \overline{\phi}_1\right)$. It is easy to check that $\underline{\phi} < \overline{\phi}_1$.

Therefore, if $\kappa/\gamma\sigma^2 Q > 1$ and $\phi \in \left(\underline{\phi}, \overline{\phi}_1\right)$, then

$$p(\phi) = m + \kappa - \frac{\gamma\sigma^2}{2}(Q + \phi) - 2\kappa\frac{Q}{Q + \phi}.$$

If $\phi < \overline{\phi}_1$, and if $p \leq m - \kappa$, i.e., $\phi \leq \underline{\phi}$, but $p + \gamma\sigma^2(Q + \phi) > m - \kappa$, the demand for the risky asset Z equals to its supply, which is given by

$$\int_{m-\kappa}^{p+\gamma\sigma^2(Q+\phi)} \frac{m_i - p}{\gamma\sigma^2} dm_i + \int_{p+\gamma\sigma^2(Q+\phi)}^{m+\kappa} (Q + \phi)\, dm_i = 2\kappa Q$$

which is simplified as

$$\left[p - (m - \kappa) + \gamma\sigma^2(Q + \phi)\right]^2 = 4\gamma\sigma^2\kappa\phi.$$

Solving for the equilibrium price $p(\phi)$, we obtain

$$p(\phi) = (m - \kappa) - \gamma\sigma^2(Q + \phi) \pm 2\sqrt{\gamma\sigma^2\kappa\phi}.$$

Observe that $p + \gamma\sigma^2(Q + \phi) > m - \kappa$ if

$$p + \gamma\sigma^2(Q + \phi) = m - \kappa \pm 2\sqrt{\gamma\sigma^2\kappa\phi} > m - \kappa,$$

i.e., $\pm 2\sqrt{\gamma\sigma^2\kappa\phi} > 0$. So we pick

$$p(\phi) = (m - \kappa) - \gamma\sigma^2(Q + \phi) + 2\sqrt{\gamma\sigma^2\kappa\phi}$$

Finally, we show that it is not possible to have $p + \gamma\sigma^2(Q + \phi) \leq m - \kappa$. Suppose not. Then, the demand for the risky asset Z equals to its supply, which is given by

$$\int_{m-\kappa}^{m+\kappa} (Q+\phi)\, dm_i = 2\kappa Q$$

But this can be true if and only if $\phi = 0$. The claim in Lemma 1 is a collection of these results according to the different ranges of the policy parameter.

Proof of Lemma 2.

If

$$\frac{m+\kappa-p}{\gamma\sigma^2} \leq Q+\phi$$

then $p = p_L^*$. Observe that this is true if

$$\phi \geq m - \left(\gamma\sigma^2+1\right) Q = \overline{\phi}_2$$

Hence, if $\kappa/\gamma\sigma^2 Q \leq 1$ and $\phi \geq m - \left(\gamma\sigma^2+1\right) Q = \overline{\phi}_2$, then $p = p_L^*$. If $\phi < \overline{\phi}_2$, can $p > m-\kappa$ be true? Borrowing from the first case,

$$p(\phi) = m+\kappa - \frac{\gamma\sigma^2}{2}(Q+\phi) - 2\kappa\frac{Q}{Q+\phi}.$$

Then $p > m-\kappa$ only if

$$f(\phi) = \left(2\kappa - \gamma\sigma^2 Q\right)\phi - \frac{\gamma\sigma^2}{2}Q^2 - \frac{\gamma\sigma^2}{2}\phi^2 > 0$$

Case II-(i). Suppose $2\kappa \leq \gamma\sigma^2 Q$. Then $f'(0) < 0$, in addition to $f(0) < 0$, $f'' < 0$, so $f(\phi) > 0$ cannot be true. Case II-(ii). Suppose $2\kappa > \gamma\sigma^2 Q$. Then the discriminant of $f(\phi)$ is

$$\begin{aligned} D &= b^2 - 4ac \\ &= 4\kappa\left(\kappa - \gamma\sigma^2 Q\right) \leq 0 \end{aligned}$$

by assumption, $\kappa/\gamma\sigma^2 Q \leq 1$, and in turn, $f(\phi) > 0$ cannot be true. Generally, we have $p \leq m-\kappa$ in the case of $\kappa/\gamma\sigma^2 Q \leq 1$. Therefore, if $\phi \leq \overline{\phi}_2$, we obtain that $p \leq m-\kappa$, but $p + \gamma\sigma^2(Q+\phi) > m-\kappa$. We conclude that if $\kappa/\gamma\sigma^2 Q \leq 1$, then

$$p(\phi) = \begin{cases} p_L^* & \phi \geq \overline{\phi}_2 \\ m-\kappa-\gamma\sigma^2(Q+\phi) + 2\sqrt{\gamma\sigma^2\kappa\phi} & \phi < \overline{\phi}_2 \end{cases}$$

□