Are Saving and Investment Cointegrated: 
A Fresh Look in Frequency Domain*

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Abstract

The standard intertemporal model of open economy with a dynamic budget constraint predicts that investment and saving should be cointegrated as long as the economy does not violate its budget constraint. Recent empirical studies of the long run investment-saving comovement for the 1947:1–1987:3 period US data, however, report conflicting findings: some studies find that the time series of investment and saving are cointegrated while others are unable to reject the null of no cointegration. One likely reason for these conflicting findings is the low power of the existing time domain cointegration tests. Using the frequency domain framework, I derive the implications of investment-saving cointegration in terms of their cross spectral characteristics. In particular, I show that if the economy’s intertemporal budget constraint is not violated and thus investment and saving are cointegrated with the cointegrating vector $[1 \ -1]$, then coherence and gain of investment and saving should equal one. Empirical examination of the US time series data for the above sample period confirms this theoretical prediction.
1. Introduction

The standard dynamic open economy model with intertemporal budget constraint (for example, Blanchard and Fischer, 1989) predicts that national saving and domestic investment should be cointegrated. The existing empirical evidence on the investment-saving cointegration, however, is mixed. Although some studies find US investment-saving time series to be cointegrated, several other studies dispute the finding. The goal of this paper is to reconcile these conflicting findings. For this I demonstrate analytically that a cointegration of time series in the time domain has testable implications for their comovement in the frequency domain. Specifically, I show that if two series are cointegrated with the cointegrating vector \([1 \ -1]\), then the zero-frequency coherence and gain of investment and saving will both equal one. When I estimate the cross-spectrum of the quarterly US series of national saving and domestic saving, I find that the zero-frequency coherence and gain of investment and saving indeed equal one. I conclude that investment and saving are cointegrated as the standard dynamic open economy model predicts.

The paper is organized as follows. In section 2, I provide theoretical background. In section 3, I analytically derive frequency domain implications of the investment-saving cointegration. In section 4, I report frequency domain estimation results. Section 5 concludes.

2. Theoretical Background

Consider a simple dynamic model of an open economy, where the central planner faces a budget constraint of the form

\[
\frac{d B_t}{dt} = \rho_t B_t + C_t + I_t + G_t - Y_t
\]

where \(B\) is foreign debt, \(C\) is consumption, \(I\) is investment, \(G\) is government expenditure, \(Y\) is output, and \(\rho_t\) is time varying world interest rate. The constraint states that an open economy may borrow from abroad to pay for excess of spending over production, or it may lend to a foreign country to accommodate excess of production over spending. Thus, the existence of world capital market enables the economy to accommodate temporary imbalances between production and spending.

Equation (1) is a nonhomogenous first order linear differential equation with a time varying coefficient. To derive the long-run implication of this budget constraint, integrate the differential equation forward by solving it for \(B_t\). This yields

\[
B_t = A \Psi^{-1} + \Psi^{-1} \int_{s}^{t} \Psi_{s} \left( Y_{s} - C_{s} - I_{s} - G_{s} \right) ds
\]
where \( A \) is an arbitrary constant which can be set to zero without affecting the main result, and
\[
\Psi_t \equiv e^{-\int_0^t \rho_s ds}
\]
is the discount factor applied to the returns of time \( t \) period into the future. Similarly,
\[
\Psi_s = e^{-\int_0^s \rho_d dv} = e^{-\int_s^{t+s} \rho_d dv},
\]
which is used in deriving (2). The discount factor \( \Psi_t^{-1} \Psi_s \) gives the time \( t \) value of a dollar to be delivered at time \( s \).

Now, assume that the \( \lim_{t \to \infty} (\Psi_t B_t) = 0 \), so that the representative agent will not be able to borrow forever. Without it, the optimal strategy is to maximize consumption by continuously borrowing and accumulating ever increasing debt. The assumption prevents the representative agent from choosing such an explosive path of borrowing. It, however, does not impede the agent’s ability to incur a temporary debt to accommodate a temporary imbalance between production and spending.

After adding and subtracting \( \int_s^T \Psi_s T_s ds \), where \( T \) is government revenue, (2) yields
\[
\int_s^T \Psi_s I_s ds = \int_s^T \Psi_s \left[ (Y_s - T_s - C_s) + (T_s - G_s) \right] ds.
\]

The first term in the brackets on the right hand side of (3) is the present discounted value of private saving, while the second term is the present discounted value of public saving. Their sum equals national saving, \( S \). Therefore,
\[
\int_s^T \Psi_s I_s ds = \int_s^T \Psi_s S_s ds.
\]
The equality in (4) says that in the long run national saving and domestic investment will be equal.

Using a version of such an open economy model, Ghosh (1995), Coakley, Kulasi, and Smith (1996), and Jansen and Schulze (1996) demonstrate this result more formally and show that if the economy’s budget constrain is satisfied, then investment and saving will be cointegrated with a cointegrating vector \([1 -1]\).

3. Frequency Domain Implications of the Investment-Saving Cointegration

Despite the theoretical predictions, the empirical evidence on the cointegration of the US national saving and domestic investment is mixed. For example, using the 1947:1–1989:4 U.S. quarterly data, Miller (1988) finds that saving and investment are cointegrated during the fixed exchange rate regime, but not during the flexible exchange rate regime. Following Feldstein and Horioka’s (1980) line of argument, he concludes that increasing capital mobility since the beginning of the 70s might have severed the long-run link between investment and saving.\(^1\) Gulley (1992) re-examines the investment-saving comovement using an improved test and finds that the levels of

\(^1\)Levy (1995) considers the investment-saving comovement under endogenous fiscal policy and reports findings similar to Miller’s for the annual U.S. time series of private saving and investment.
saving and investment are not cointegrated in either periods. Similarly, Otto and Wirjanto (1989) conclude that national saving and investment in the U.S. are not cointegrated. These studies use time domain cointegration tests such as the ones proposed by Engle and Granger (1987) or Johansen (1988), which rely on the standard Dickey-Fuller (DF) and Augmented Dickey-Fuller (ADF) unit root tests, whose low power is widely recognized.\textsuperscript{2}

To resolve the conflicting findings, let us examine the frequency domain implications of the cointegration of the time series of investment and saving. Assume that the time series of investment and saving are difference stationary. Thus, let both $I_t$ and $S_t$ be $I(1)$, so that they can be written as $I_t = I_{t-1} + u_t$ and $S_t = S_{t-1} + v_t$, respectively, where $u_t \sim I(0)$, and $v_t \sim I(0)$. Moreover, let us assume that investment and saving are cointegrated with the cointegration vector $[1 -1]$. Then, investment and saving processes share a common stochastic trend and therefore, can be written in a matrix notation

$$
\begin{bmatrix}
I_t \\
S_t
\end{bmatrix} = \begin{bmatrix}
T_t + i_t \\
T_t + s_t
\end{bmatrix}.
$$

(5)

where $T_t$ is the common stochastic trend with the property $(1 - L)T_t = z_t$, $z_t \sim \text{iid } 0, \sigma_z^2$ is a white noise process, $i_t \sim I(0)$, and $s_t \sim I(0)$. Applying difference operator, $\Delta = (1 - L)$, to $(5)$ yields a bivariate stationary process

$$
\begin{bmatrix}
(1 - L) I_t \\
(1 - L) S_t
\end{bmatrix} = \begin{bmatrix}
z_t + (1 - L)i_t \\
z_t + (1 - L)s_t
\end{bmatrix}.
$$

(6)

with the spectral matrix

$$
f(\omega) = \begin{bmatrix}
\hat{f}_I & \hat{f}_{I,\Delta S} \\
\hat{f}_{S,\Delta I} & \hat{f}_S
\end{bmatrix}.
$$

(7)

The elements on the diagonal of the $f(\omega)$ matrix are the spectral density functions of $(1 - L)I_t$ and

\textsuperscript{2}It is well-known that the standard time series cointegration tests have low power and the differences in the power properties of the different tests used by the above studies may account for the conflicting findings they report. Another, less frequently discussed, drawback of these tests, however, is that their null and the alternative hypotheses are formulated in such a way that they are biased against the cointegration hypotheses. To see this, recall that in the standard unit root tests such as the DF or the ADF tests, the null hypothesis is that there is a unit root, and this hypothesis is maintained unless there is overwhelming evidence against it. In the cointegration test the null hypothesis is \textit{no cointegration}, and this null hypothesis is maintained unless there is overwhelming evidence against it. Therefore, these unit root/cointegration tests are in favor of unit root and no cointegration. As Maddala (1991) points out, this asymmetry does not arise in the Bayesian setting, since there the null and the alternative hypotheses are on the same footing. Perhaps this is the reason why many series that have been identified as difference-stationary when standard unit root tests are used, are found to be trend-stationary within Bayesian framework.
\[(1 - L)S_t, \text{ defined by} \]
\[
\delta_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta i}(\tau) e^{-i\tau\omega} d\tau
\]
\[
\delta_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta s}(\tau) e^{-i\tau\omega} d\tau,
\]
where \(\gamma_{\Delta i}(\tau)\) and \(\gamma_{\Delta s}(\tau)\) are the autocovariance functions of \((1 - L)I_t\) and \((1 - L)S_t\), respectively.

The off-diagonal elements of the \(f(\omega)\) matrix are the cross spectral density functions of \((1 - L)I_t\) and \((1 - L)S_t\), defined by
\[
\delta_{i\Delta s} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta i\Delta s}(\tau) e^{-i\tau\omega} d\tau
\]
and
\[
\delta_{s\Delta i} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \gamma_{\Delta s\Delta i}(\tau) e^{-i\tau\omega} d\tau,
\]
where \(\gamma_{\Delta i\Delta s}(\tau)\) and \(\gamma_{\Delta s\Delta i}(\tau)\) are the cross covariance functions of \((1 - L)I_t\) and \((1 - L)S_t\), and \((1 - L)S_t\) and \((1 - L)I_t\), respectively.

To compute the elements of the \(f(\omega)\) matrix, I first compute the autocovariance-crosscovariance matrix of (6). It is given by
\[
\begin{bmatrix}
\gamma_{\Delta i}(\tau) & \gamma_{\Delta i\Delta s}(\tau) \\
\gamma_{\Delta s\Delta i}(\tau) & \gamma_{\Delta s}(\tau)
\end{bmatrix}
\]
\[
= E \left[ \begin{bmatrix}
z_t + \tau + \Delta i_t + \tau \\
z_t + \tau + \Delta s_t + \tau
\end{bmatrix} \begin{bmatrix}
z_t + \Delta i_t \\
z_t + \Delta s_t
\end{bmatrix} \right]
\]
\[
= \begin{bmatrix}
\gamma_z(\tau) + \gamma_i(\tau) + \gamma_{z1}(\tau) + \gamma_{i1}(\tau) & \gamma_z(\tau) + \gamma_{12}(\tau) + \gamma_{12}(\tau) + \gamma_{12}(\tau) \\
\gamma_z(\tau) + \gamma_{2z}(\tau) + \gamma_{z1}(\tau) + \gamma_{21}(\tau) & \gamma_z(\tau) + \gamma_{21}(\tau) + \gamma_{12}(\tau) + \gamma_{2z}(\tau)
\end{bmatrix},
\]
where subscripts 1 and 2 denote \(\Delta i\) and \(\Delta s\), respectively, for notational simplicity, and the diagonal and the off-diagonal elements are the autocovariance and the crosscovariance functions, respectively.

Applying Fourier transform to both sides of equation (10), multiplying through by \(\frac{1}{2\pi}\), and using the spectrum and cross-spectrum definitions provided by (8a)–(8b) and (9a)–(9b), we get the spectral matrix
\[
f(\omega) = \begin{bmatrix}
f_i(\omega) + f_i(\omega) + f_i(\omega) + f_i(\omega) & f_1(\omega) + f_1(\omega) + f_1(\omega) + f_1(\omega) \\
f_1(\omega) + f_{12}(\omega) + f_{12}(\omega) + f_{12}(\omega) & f_1(\omega) + f_1(\omega) + f_1(\omega) + f_1(\omega)
\end{bmatrix},
\]
which can be rewritten as
\[
f(\omega) = \begin{bmatrix}
    f(\omega) & f(\omega) \\
    f(\omega) & f(\omega)
\end{bmatrix} + \begin{bmatrix}
    \bar{f}(\omega) + \bar{f_i}(\omega) + \bar{f_z}(\omega) & \bar{f}(\omega) + \bar{f_z}(\omega) + \bar{f_z}(\omega) \\
    \bar{f}(\omega) + \bar{f_i}(\omega) + \bar{f_z}(\omega) & \bar{f}(\omega) + \bar{f_z}(\omega) + \bar{f_z}(\omega)
\end{bmatrix}.
\] (12)

The cross spectrum terms in (12) can be written in cartesian form because the spectral matrix \(f(\omega)\) is in general a complex valued function. Thus, for example, we can write \(f_{x,1}(\omega) = c_{x,1}(\omega) - iq_{x,1}(\omega)\), where \(c\) denotes the cospectral density function and \(q\) denotes the quadrature spectral function.

Therefore, using Priestley’s (1981, p. 668, Equation 9.1.53) result that \(f_{y,x}(\omega) = f_{x,y}(\omega)\), (12) can be rewritten as

\[
f(\omega) = \begin{bmatrix}
    f(\omega) & f(\omega) \\
    f(\omega) & f(\omega)
\end{bmatrix} + \begin{bmatrix}
    \bar{f}(\omega) + \bar{f_i}(\omega) + \bar{f_z}(\omega) & \bar{f}(\omega) + \bar{f_z}(\omega) + \bar{f_z}(\omega) \\
    \bar{f}(\omega) + \bar{f_i}(\omega) + \bar{f_z}(\omega) & \bar{f}(\omega) + \bar{f_z}(\omega) + \bar{f_z}(\omega)
\end{bmatrix}
\] (13)

where \(\bar{\cdot}\) denotes complex conjugate. Combining (13) with cartesian representation of \(f_1(\omega)\) and \(f_2(\omega)\), yields

\[
f(\omega) = \begin{bmatrix}
    f(\omega) & f(\omega) \\
    f(\omega) & f(\omega)
\end{bmatrix} + \begin{bmatrix}
    \bar{f}(\omega) + 2c_z(\omega) & \bar{f}(\omega) + 2c_z(\omega) \\
    \bar{f}(\omega) + 2c_z(\omega) & \bar{f}(\omega) + 2c_z(\omega)
\end{bmatrix}
\] (14)

Now, consider the value of the spectral matrix \(f(\omega)\) at frequency \(\omega = 0\), which using (7) can be written as

\[
f(0) = \begin{bmatrix}
    \hat{f}(0) & \hat{f}_{\Delta_1}(0) \\
    \hat{f}_{\Delta_1}(0) & \hat{f}(0)
\end{bmatrix}.
\] (15)

Recall that \(z_t\) is a white noise process, and therefore its theoretical spectrum is flat and equals

\[
\hat{f}(\omega) = \frac{\sigma_z^2}{2\pi}
\]

for all frequencies \(-\pi \leq \omega \leq \pi\). In addition, \(\Delta 1\) and \(\Delta s\) are \((−1)\), and therefore their zero-frequency spectral density, cross spectral density, and cospectral density functions equal zero. Thus, every element of the second matrix of the right hand side of (14) vanishes, and therefore the spectral matrix (10), evaluated at frequency \(\omega = 0\), becomes

\[
f(0) = \begin{bmatrix}
    \sigma_z^2 & \sigma_z^2 \\
    \sigma_z^2 & \sigma_z^2
\end{bmatrix}
\] (16)

To see the implications of this result for the behavior of the theoretical squared coherence and gain, recall from polar representation of \(f(\omega)\) that
where $K_{ΔIΔS}(ω)$ and $Γ_{ΔIΔS}(ω)$ denote the squared coherence and the gain of saving and investment, respectively (Jenkins and Watts, 1968). Then, using the matrix (16) along with the definitions of squared coherence and gain provided in (17) and (18), we get that at the zero frequency

\begin{align*}
K_{ΔIΔS}(ω) &= \frac{|f_{ΔIΔS}(ω)|^2}{f_{ΔI}(ω)f_{ΔS}(ω)} = \frac{c_{ΔIΔS}(ω)^2 + q_{ΔIΔS}(ω)^2}{f_{ΔI}(ω)f_{ΔS}(ω)},
\end{align*}

and

\begin{align*}
Γ_{ΔIΔS}(ω) &= \frac{|f_{ΔIΔS}(ω)|}{f_{ΔS}(ω)} = \frac{\left[ c_{ΔIΔS}(ω)^2 + q_{ΔIΔS}(ω)^2 \right]^{1/2}}{f_{ΔS}(ω)},
\end{align*}

Thus, we find that if investment and saving are cointegrated with the cointegration vector $[1 -1]$, then the zero frequency squared coherence and gain of investment and saving will both equal one.

4. Frequency Domain Estimation Results

To examine whether the frequency domain behavior of the investment-saving comovement is consistent with their cointegration, I estimate the cross-spectrum of investment and saving. For this I estimate and plot squared coherence, phase, and gain. Squared coherence is analogous to the square of correlation coefficient and measures the degree to which one series can be represented as a linear function of the other. Phase measures the phase difference or the timing (i.e., lead or lag) between the frequency components of the two series. Gain indicates how much the spectrum of one series has been amplified to approximate the corresponding frequency component of the other. It is essentially the regression coefficient of one series on another at frequency $ω$. Along with point estimates of these quantities, I also report their statistical significance: as a significance test for squared coherence, I test whether it equals zero; for phase and gain, I provide 95% confidence interval.

I used investment and saving measured in levels as well as in rates (as a fraction of output). Unit root test results (not reported) indicate difference-stationarity of all series for all samples examined. Therefore, all series were differenced before applying the frequency domain analysis. Whether measured in levels or in rates, the results turned out to be almost identical. Therefore, I only present the findings, displayed in Figures 1–3, for levels. Panel (a) of each figure displays the
squared coherence, panel (b) displays the phase, and panel (c) displays the gain. On the figures, the vertical lines divide the frequency axis into the long-run ($0 \leq \omega < 0.19$, 32-quarter cycles or longer), the business cycle ($0.19 \leq \omega < 0.52$, 12–32 quarter cycles), and the short-run frequency bands ($0.52 \leq \omega \leq \pi$, 2–12 quarter cycles).

The entire quarterly data set I use covers the 1947:1–89:4 period, which was selected to match the sample period used by Miller (1988), Otto and Wirjanto (1989), and Gulley (1992). I also provide results for two subperiods: 1947:1–71:2, which is a period of fixed exchange rate regime, and 1971:3–89:4, which is a period of flexible exchange rate regime. As Figures 1–3 suggest, the estimates of the squared coherence, phase, and gain exhibit a very similar behavior across the three sample periods examined. All three coherences indicate very high correlation between investment and saving over the entire frequency band $[0, \pi]$ with the exception of the frequencies $\omega \approx 1.572$ and $\omega \approx \pi$. These frequencies correspond to 4-quarter and 2-quarter cycles, respectively, and therefore are considered seasonal frequencies. The zero-frequency squared coherence is near one, as predicted by the theoretical results derived in the previous section.\(^3\) The estimated phases are flat and statistically indifferent from zero at all frequencies. This suggests that the time series of investment and saving are in phase and move contemporaneously together regardless of the time horizon. The estimated gain statistically is not different from one for most frequencies with the exception of perhaps very short-run frequencies close to $\omega \approx \pi$. There seems to be no substantial difference in the behavior of the gain for the fixed and flexible exchange rate regime subperiods. More importantly, the zero frequency gain is in the neighborhood of one as the theoretical model predicts.

5. Conclusion

The standard dynamic model of open economy with intertemporal budget constraint predicts that investment and saving are cointegrated. Recent empirical studies of the long run investment-saving comovement for the 1947:1–1987:3 period US data, however, report conflicting findings: some studies find that the time series of investment and saving are cointegrated while others are unable to reject the null of no cointegration. Using the frequency domain framework, I derive the implications of investment-saving cointegration in terms of their cross spectral characteristics. Specifically, I show that if the economy’s intertemporal budget constraint is not violated and thus investment and saving are cointegrated with the cointegrating vector $[1 \ -1]$, then coherence and gain of investment and saving should equal one. Empirical examination of the US time series data for the

\(^3\)Note that there is a conceptual difficulty with testing formally the null hypothesis $H_0: K_{\Delta I, \Delta S}(\omega) = 1$ with finite data because $K_{\Delta I, \Delta S}(\omega) \leq 1$. (Recall that squared coherence is similar to the correlation coefficient and therefore cannot exceed one.) Therefore, it will be close to impossible to make finite data set to statistically “confess” that the squared coherence equals 1. See Amos and Koopmans (1963).
above sample period confirms this theoretical prediction. Thus, the findings indicate that the series are indeed cointegrated which suggests that the U.S. economy is solvent in the sense that it does not seem to violate its dynamic budget constraint.
References


Maddala, G.S., 1991, Of what use are cointegration tests? manuscript, University of Florida.


Figure 1. Estimated Cross-Spectral Density of National Saving and Domestic Investment, US, 1947:1–1989:4
Figure 2. Estimated Cross-Spectral Density of National Saving and Domestic Investment, US, 1947:1–1971:2
Figure 3. Estimated Cross-Spectral Density of National Saving and Domestic Investment, US, 1971:3–1989:4