Option Prices, Preferences, and State Variables

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Abstract

This paper surveys recent developments in the theory of option pricing. The emphasis is on the interplay between option prices and investors’ impatience and their aversion to risk. The traditional view, steeped in the risk-neutral approach to derivative pricing, has been that these preferences play no role in the determination of option prices. However, the usual lognormality assumption required to obtain preference-free option pricing formulas is at odds with the empirical properties of financial assets. The lognormality assumption is easily reconcilable with those properties by the introduction of a latent state variable whose values can be interpreted as the states of the economy. The presence of a covariance risk with the state variable makes option prices depend explicitly on preferences. Generalized option pricing formulas, in which preferences matter, can explain several well-known empirical biases associated with preference-free models such as that of Black and Scholes (1973) and the stochastic volatility extensions of Hull and White (1987) and Heston (1993).
1. Introduction

The stochastic discount factor (SDF) or pricing kernel model provides a general framework for asset pricing. Indeed, most existing asset pricing models can be cast in the SDF framework. This includes the capital asset pricing model (CAPM) of Sharpe (1964), Linter (1965) and Black (1972), the intertemporal general equilibrium consumption-based capital asset pricing model (CCAPM) of Rubinstein (1976) and Lucas (1978), and even the Black and Scholes (1973) (BS) option pricing model. The SDF framework emphasizes the structure placed on financial asset prices and payoffs by the assumption that asset markets do not permit the presence of arbitrage opportunities—transactions involving no cash outlay that result in a sure profit. In the absence of arbitrage opportunities, market prices can be expressed by the expected-value relation:

\[ p_t = E_t[m_{t+1}g_{t+1}], \]

where \( p_t \) is the asset price, \( g_{t+1} \) is the asset’s future payoff, and \( m_{t+1} \) is an appropriate SDF. The SDF is a random variable which can only be positive. It generalizes the notion of a discount factor to a world of uncertainty about future states of nature. While market completeness ensures that the SDF is unique, the no-arbitrage condition is sufficient to guarantee its existence. The basic asset pricing equation can also be written as

\[ 1 = E_t[m_{t+1}r_{t+1}], \]

where \( r_{t+1} = g_{t+1}/p_t \) defines gross returns. Gross returns can be defined either in nominal or real terms; correspondingly, the SDF must then be expressed in nominal or real terms.

The basic asset pricing equation can be seen as an application of the Arrow-Debreu model, where a state price exists for each state of nature and the market price of any financial asset is just the sum of its possible future payoffs, weighted by the appropriate state prices; see Varian (1987) for details.

Equilibrium models with optimizing investors imply tight links between the SDF and the marginal utilities of investors’ consumption. The SDF can therefore be used to relate asset prices to the underlying preferences of investors. In consumption-based asset pricing models, a single representative investor is assumed to derive utility from the aggregate consumption of the economy. The first-order condition or Euler equation describing the representative investor’s optimal consumption and portfolio plan is

\[ p_t u'(c_t) = E_t[\beta u'(c_{t+1})g_{t+1}] \]
or
\[ p_t = E_t \left[ \frac{\beta u'(c_{t+1})}{u'(c_t) \cdot g_{t+1}} \right], \]
where \( \beta \) is a subjective discount factor capturing the investor’s impatience, and \( u'(c) \) denotes the marginal utility of consumption, \( c \); the curvature of the utility function generates aversion to risk and to intertemporal substitution. The term \( p_t u'(c_t) \) in the Euler equation is the marginal utility cost that results from the purchase of one additional unit of the asset at time \( t \), while \( E_t [\beta u'(c_{t+1})g_{t+1}] \) is the expected marginal utility benefit obtained from the asset’s payoff at time \( t + 1 \). An optimizing investor takes a position in the asset such that the marginal cost equals the marginal benefit. Here the SDF is the intertemporal marginal rate of substitution—the discounted ratio of marginal utilities in two successive periods—of the representative investor; i.e., \( m_{t+1} = \beta u'(c_{t+1})/u'(c_t) \).

Since the seminal work of Hansen and Singleton (1982, 1983), the power utility function:
\[ u(c_t) = c_t^{1-\gamma} - 1 \]
where \( \gamma \) is the coefficient of relative risk aversion, has commonly been used in consumption-based asset pricing models. When utility is of this power form, the elasticity of intertemporal substitution is given by the reciprocal of the coefficient of relative risk aversion. In this case, marginal utility is \( u'(c_t) = c_t^{-\gamma} \) and the resulting pricing equation:
\[ 1 = E_t \left[ r_{t+1} \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \]
can be used to estimate the coefficient of relative risk aversion. Formal statistical tests, however, tend to reject the the consumption-based asset pricing model with power utility. In particular, this model implies a very large coefficient of relative risk aversion given postwar U.S. stock returns and consumption data; see Grossman and Shiller (1981) and Hansen and Singleton (1983). This is the well-known equity premium puzzle of Mehra and Prescott (1985). In Section 6 we present a more flexible utility specification, due to Epstein and Zin (1989), that retains many of the attractive features of power utility but relaxes the tight link between the coefficient of relative risk aversion and the elasticity of intertemporal substitution.

Empirical work on the estimation of preference parameters has typically focused on stock returns and not so much on derivative securities—financial securities whose payoffs depend on the prices of other securities. Yet, under the no-arbitrage condition, the basic
pricing equation, \( p_t = E_t[m_{t+1}g_{t+1}] \), is valid for any financial asset, including for example European call options with payoff function \( g_{t+1} = \max[0, S_{t+1} - K] \), where \( K \) is the exercise or strike price. If one assumes that the SDF and asset returns are jointly lognormal, then the option price is given by the well-known BS formula. The no-arbitrage condition yields not only the option’s price but also the means to replicate exactly the option’s payoff by a particular dynamic investment strategy involving only the underlying stock and risk-free bond. In this case, the relevant characteristics of the SDF that capture investors’ preferences do not explicitly enter the option pricing formula except through the bond price and the underlying asset price; i.e., the option pricing formula is preference free. Thus, in a BS world, option prices would not be any more informative about preferences than bonds and stock returns.

The lognormality assumption, however, is at odds with the conditional skewness and excess kurtosis observed in time series of stock returns. Since the seminal paper of Clark (1973), a versatile way to reconcile the lognormality assumption with the time-series properties of financial price changes is to introduce mixture components; i.e., to assume that the stochastic discount factor and asset returns are jointly lognormal conditional on a latent state variable. In the equilibrium-based models of Cecchetti, Lam, and Mark (1990, 1993) and Bonomo and Garcia (1994a, b, 1996), the state variable governs the evolution of the fundamentals—consumption and dividends—of the economy. This variable captures the states of the economy which are typically represented by low consumption growth associated with high volatility of dividend growth, or by high consumption growth together with a low volatility of dividend growth. A contemporaneous correlation or leverage effect between the state variable and the fundamentals makes the preference parameters play an additional role in the pricing of options above and beyond their impact on bond and stock prices.

The rest of this paper expounds two main points. The first is that preferences matter for option pricing, and the second is the converse proposition that option prices are informative about preferences. The paper is organized as follows. Section 2 shows how preferences get buried into the bond and stock prices when pricing derivative securities under the usual lognormality assumption. Section 3 shows how the role of preferences in option prices can be uncovered by the introduction of a latent state variable. Section 4 discusses the intimate relations between option prices, state-price densities (the generalization of Arrow-Debreu prices to a continuum of states of nature), and the volatilities implied by the BS formula given actual market option prices. Section 5 discusses various methods that have been
proposed to extract information about preferences from option prices in a nonparametric fashion. Section 6 outlines an intertemporal equilibrium option pricing model that features latent state variables. Section 7 concludes.

2. Preferences Buried by Lognormal Discounting

The fundamental equation of asset pricing theory shows that the price, \( p_t \), of an asset at time \( t \) with future random payoff, \( g_{t+1} \), denoted \( p_t = p_t(g_{t+1}) \), can be written as

\[
p_t(g_{t+1}) = E_t[m_{t+1}g_{t+1} | J_t] = E_t[m_{t+1}g_{t+1}],
\]

where \( E_t[\cdot] \) is the expectations operator conditional on time-\( t \) information, \( J_t \). In general, the payoff of an asset is a random variable whose future value cannot be known with certainty at the time of pricing. A risk-free asset is one whose future payoff is known at time \( t \). Whether an asset’s payoff is random or not, its price can always be represented by Equation (1).

Denote by \( B(t, t+1) \) the time-\( t \) price of a pure discount bond that pays one dollar regardless of the state of nature at time \( t + 1 \). By Equation (1), its price is given by

\[
B(t, t+1) = E_t[m_{t+1}].
\]

Given (2) and that \( E_t[m_{t+1}g_{t+1}] = E_t[m_{t+1}]E_t[g_{t+1}] + Cov_t[m_{t+1}, g_{t+1}] \), the asset price can be written as

\[
p_t(g_{t+1}) = B(t, t + 1)E_t[g_{t+1}] + Cov_t[m_{t+1}, g_{t+1}],
\]

which shows that the asset’s riskiness is determined by the covariance of its payoff with the SDF. The first term appearing on the right-hand side of (3) is the expected payoff discounted at a risk-free rate. This is the asset’s price in a risk-neutral world where investors would not require a premium to hold a risky asset whose payoff covaries with the SDF. The second term on the right-hand side of (3) is the compensation risk-averse investors require for taking on risk: an asset whose payoff covaries positively with the SDF has its price raised and vice versa.

Denote by \( h_{t+1} = h(J_t, S_{t+1}) = h_t(S_{t+1}) \) the payoff of a derivative security whose value is contingent upon the value of the stock at time \( t + 1, S_{t+1} \). The most common risk-neutral approach to price derivative securities amounts to rewriting the fundamental pricing equation as

\[
p_t(h_{t+1}) = E_t[m_{t+1}h(S_{t+1})] = B(t, t + 1)E_t^N[h(S_{t+1})],
\]
where $E^N_t[\cdot]$ is the conditional expectations operator with respect to a risk-neutral distribution—one obtained by multiplying the density function of the objective distribution by $m_{t+1}/B(t, t+1)$. In risk-neutral form, the derivative asset is priced as if investor preferences were risk neutral; i.e., under risk neutrality, the derivative price is the discounted expected payoff under the risk-neutral distribution without any risk compensation. However, preferences get buried by risk-neutral discounting only insofar as the risk-neutral expectations operator can be fully characterized from the observation of bond and stock prices, without further references to any preference characteristics.

The so-called risk neutral valuation relationship (RNVR) strategy put forward by Brennan (1979) focuses on nice cases where the change of probability measure needed to compute risk-neutral expectations is akin to a simple transformation of the underlying asset price through:

$$p_t(h_{t+1}) = B(t, t + 1)E^N_t[h_t(S_{t+1})] = B(t, t + 1)E_t[h_t(S^N_{t+1})],$$

for some suitably defined risk-adjusted value, $S^N_{t+1}$. If one wants to get such a pricing formula for any function $h$, it must be true in particular for linear functions, and thus:

$$B(t, t + 1)E_t[S^N_{t+1}] = S_t.$$

Brennan’s proposal is to characterize the cases where this may be obtained by defining $S^N_{t+1}$ as a simple rescaling of $S_{t+1}$:

$$B(t, t + 1)S^N_{t+1} = [S_{t+1}/E_tS_{t+1}]S_t.$$

In the RNVR form given by (5), the price of the derivative security no longer depends explicitly on preference parameters. Investors’ preferences are buried, first in the price $B(t, t + 1)$ of the pure discount bond, and second in the price of the underlying asset through the risk premium $B(t, t + 1)E_tS_{t+1}/S_t$.

By focusing on the SDF provided by a time-separable utility function of a representative investor, $m_{t+1} = \beta u'(C_{t+1})/u'(C_t)$, with the additional assumption that $[\log(C_{t+1}), \log(S_{t+1})]$ is jointly normal conditional on $J_t$, Brennan shows that a necessary and sufficient condition for RNVR pricing is that the marginal utility function, $u'$, is a power function. However, one should note that the role of the power utility function is to obtain joint conditional normality of $[\log(u'(C_{t+1})), \log(S_{t+1})]$ from that of $[\log(C_{t+1}), \log(S_{t+1})]$. In other words, the crucial assumption to obtain RNVR pricing in this one-period setting is

**Assumption 2.1.** Conditional on $J_t$, $[\log(m_{t+1}), \log(S_{t+1}/S_t)]$ is normally distributed.
The option pricing literature contains numerous well-known examples of models involving Assumption 2.1. When one explicitly writes the SDF corresponding to a one-period BS option pricing model, one realizes (see Buraschi and Jackwerth 2001) that \(\log(m_{t+1})\) is an affine function of the normal log-return, \(\log(S_{t+1}/S_t)\). Therefore, Assumption 2.1 is fulfilled in the BS case with a degenerate joint normal distribution. Moreover, as Heston and Nandi (2000) note, the GARCH option pricing model of Duan (1995) assumes that the value of a call option one period prior to expiration obeys the BS formula. Hence, GARCH option pricing is also based on a degenerate conditional normal probability distribution of \([\log(m_{t+1}), \log(S_{t+1}/S_t)]\) given \(J_t\). The only difference with the BS geometric random walk is that the conditional variance of \(\log(S_{t+1}/S_t)\) given \(J_t\) will now depend explicitly on \(J_t\).

We can conclude that all the aforementioned option pricing models are indeed simple applications of RNVR pricing since Assumption 2.1 implies an RNVR. To prove this statement, it suffices to extend the proof in Brennan (1979) using the SDF formulation and Girsanov’s theorem (which makes clear the analogy with option pricing in the context of diffusion models; see Renault 2001). An elementary way to achieve this is to realize that when two variables \((X, Y)\) are jointly normally distributed, then

\[
E[\exp(X)h(Y)] = E[\exp(X)]E[h(Y + Cov(X,Y))].
\]  

(6)

Application of (6) to \(X = \log(m_{t+1})\) and \(Y = \log(S_{t+1})\) gives the announced crucial result that the underlying asset price can be conveniently rescaled so as to obtain an RNVR under Assumption 2.1.

For example, with \(h_{t+1} = h(S_{t+1}) = \max[0, S_{t+1} - K]\) —the payoff of a European call—we get

\[
\pi_t(h_{t+1}) = B(t, t+1)E_t \max[0, S_{t+1}^N - K],
\]

where

\[
\log(S_{t+1}^N/S_t) = (S_{t+1}/S_t) - \log E_t(S_{t+1}B(t, t+1)/S_t).
\]

In the BS world, \(\log(S_{t+1}/S_t)\) follows a normal distribution with mean \(\log(E_t(S_{t+1}/S_t)) - \sigma^2/2\) and variance \(\sigma^2\). Then \(\log(S_{t+1}^N/S_t)\) follows a normal distribution with mean \(-\log B(t, t+1)\) and variance \(\sigma^2/2\).

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2Garcia and Renault (1998a) and Kallsen and Taqqu (1998) discuss in further detail the maintained assumptions of GARCH option pricing both in terms of equilibrium and no arbitrage.

3The result in (6) is obtained by noting that: (i) \(X = E(X) + b[Y - E(Y)] + U\), where \(b = Cov[X,Y]/Var(Y)\) and \(U\) has mean zero and is independent of \(Y\), and (ii) \(E[\exp(b[Y - E(Y)]) - bCov(X,Y)/2|h(Y)] = E[h(Y + Cov(X,Y))],\) by a simple plugging of the exponential into the normal density.
1) \(-\sigma^2/2\) and variance \(\sigma^2\). Straightforward computation of expectations with normal distributions gives the BS option pricing formula:

\[
\pi_t(h_{t+1}) = BS[S_t, K, \sigma^2],
\]

\[
BS[S_t, K, \sigma^2] = S_t\Phi(d_1) - KB(t, t+1)\Phi(d_2),
\]

where \(d_1 = (1/\sigma)\log(S_t/(KB(t, t+1))) + \sigma/2\) and \(d_2 = d_1 - \sigma\).

Generally speaking, preference-free option pricing can be obtained within two alternative settings: either a linear factor model with an affine regression of \(m_{t+1}\) on \(S_{t+1}\) for the conditional expectation \(E_t[m_{t+1} | S_{t+1}]\) given either a normality or a multinomial assumption, or a log-linear factor model with an affine regression of \(\log(m_{t+1})\) on \(\log(S_{t+1})\) for the conditional expectation \(E_t[\log m_{t+1} | S_{t+1}]\). The basic intuition is that, without such a linearity property of the conditional expectation, the price of contingent claims with payoffs that are nonlinear functions of the underlying asset payoff cannot be straightforwardly deduced from the price of the underlying asset. Even though it is worth stressing that the aforementioned lognormality properties are only conditional, and therefore do not preclude conditional heteroskedasticity and unconditional leptokurticity of asset returns (as in the GARCH option pricing model), it is still empirically relevant to think of a way to relax them. Since the seminal paper of Clark (1973), a versatile tool to relax normality of asset (log) returns is to introduce state variables. This approach also has the advantage that it may be used to link asset prices to the states of the business cycle.

3. Preferences Uncovered by State Variables

In this section we further explore the log-linear model but with a state variable added to the conditioning information set.

Assumption 3.1. There exists a latent state variable, \(U_{t+1}\), such that, conditional on \(J_t\) and \(U_{t+1}\), \([\log m_{t+1}, \log(S_{t+1}/S_t)]\) is normally distributed.

Hence, the conditional probability distribution of \([\log(m_{t+1}), \log(S_{t+1}/S_t)]\) given \(J_t\) is a mixture of normals with mixture component \(U_{t+1}\). The mixture model is quite standard and contains as particular cases the widely-used stochastic volatility model first proposed by Taylor (1986) and used for option pricing by Hull and White (1987) and Heston (1993).

\(^4\)Amin and Ng (1993) use a fairly similar framework in discrete time, except that in their paper the SDF is specified from a time separable utility function.
In contrast with some standard option pricing models under stochastic volatility, we have to assume here, in order to relax RNVR pricing, that the realizations of the state variable are not instantaneously observable by the investors at the beginning of period $t$. It is only at the end of the period that investors learn the realized value; i.e., $U_{t+1} \in J_{t+1}$. This information structure can be rationalized by the genuine uncertainty and learning about business cycle fluctuations and the actual state of the economy considering that the discrete-time intervals are somewhat arbitrary and can be infinitely divided.

In this framework, the pricing formula becomes

$$p_t(h_{t+1}) = E_t \{ E_t[m_{t+1}h_t(S_{t+1}) \mid U_{t+1}] \}$$

or, more explicitly, by exploiting the Girsanov-type formula put forward in the previous section:

$$p_t(h_{t+1}) = E_t \{ E_t[m_{t+1} \mid : U_{t+1}]E_t[h_t(S_{t+1}) \exp(Cov_t[\log(m_{t+1}), \log(S_{t+1}/S_t) \mid U_{t+1}]) \mid U_{t+1}] \} .$$

Using the bond and stock pricing equations, we arrive at a generalized RNVR formula:

$$p_t[h_{t+1}] = E_t \left\{ B^N(t, t+1)E_t[h_t(S_{t+1}^N \cdot Q(t, t+1)) \mid U_{t+1}] \right\} ,$$

where $S_{t+1}^N$ is a rescaled value of the underlying stock price defined by

$$B^N(t, t+1)S_{t+1}^N = S_t \frac{S_{t+1}}{E_t[S_{t+1}]} ,$$

with $B^N(t, t+1) = E_t[m_{t+1} \mid U_{t+1}]$ and $Q(t, t+1) = E_t[m_{t+1}(S_{t+1}/S_t) \mid U_{t+1}]$. Note that $B^N(t, t+1)$ and $S_tQ(t, t+1)$ can be interpreted respectively as the bond and stock price in the fictitious world where the state variable $U_{t+1}$ is known by investors at the beginning of period $t$. By the law of iterated expectations, the time-$t$ expected value of these prices coincide with the actual prices; i.e., $B(t, t+1) = E_t[B^N(t, t+1)]$ and $E_t[Q(t, t+1)] = 1$.

The pricing formula (7) thus generalizes the basic RNVR formula by the introduction of bond and stock price functions that depend on the state variable. These functions will coincide with the observed bond and stock prices if and only if there is no mixing effect in the joint probability distribution of the SDF and the stock return. Otherwise, the presence of a covariance risk with the state variable makes the price of derivative securities depend explicitly on preferences through the occurrence of the SDF in the definition of $B^N(t, t+1)$ and $Q(t, t+1)$.

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4. Option Prices, State-Price Densities, and Volatility Smiles

In applying option pricing models, practitioners encounter the difficulty that some of the parameters may be unobservable. In the benchmark BS model, the volatility of the underlying stock price is not directly observed and therefore needs to be “estimated”. While there are several statistical methods to estimate volatility, an alternative and common approach consists of extracting the volatility implied by an option price observed in the market. With this methodology, option market practitioners are attempting to answer the following question posed by Sutton (2000): is it possible to find economic models that work? See Renault (2002) for more discussion. In this section, we propose to go even further by pointing out the tight relationship between a practical tool and a sophisticated economic concept. On the practical side, there is the so-called volatility smile used by option market practitioners as a measure of discrepancy between actual option prices and those in the ideal BS world. The volatility smile characterizes, at a given date and for a given maturity, option prices (and more precisely the corresponding BS implied volatilities) as a function of the option (log) exercise price. On the theoretical side, the Arrow-Debreu security prices tell us, as a function of the possible states of nature, how much investors are ready to pay today in order to get one dollar at a given future date contingently upon the realization of a particular state of nature. The main message of this section is that the two sets of figures basically convey the same information about investors’ preferences.

In order to demonstrate this, consider now the fundamental pricing equation in a multi-period context:

$$p_t(g_T) = E_t[m_{t,T}g_T],$$

where the time-$t$ price of a payoff occurring at time $T > t$ is characterized by the multi-period stochastic discount factor $m_{t,T}$. The multi-period SDF can be obtained by chaining together one-period SDFs and finding $m_{t,T} = m_{t,t+1}m_{t+1,t+2}...m_{T-1,T}$; the multi-period SDF can thus only assume positive values. This positivity property allows one to define a risk-neutral probability measure, $N_{t,T}$, as the original objective probability density function multiplied by $m_{t,T}/E_t[m_{t,T}]$ so that the multi-period asset pricing equation can be rewritten as

$$p_t(g_T) = B(t,T)E_t^{N_{t,T}}[g_T],$$

where $B(t,T) = E_t[m_{t,T}]$ is now the price of a pure discount bond maturing at time $T$;
the notation $E^{N_t,T}_t[\cdot]$ indicates that the expectation is computed under the multi-period (forward) risk-neutral probability measure.

By using the risk-neutral probabilities, assets are priced as if investors are all risk neutral over the period $[t, T]$. This means that given the risk-neutral probability density function, any asset with a single liquidating time-$T$ payoff can be priced at time $t$ using Equation (8). The risk-neutral probability distribution, also known as the state-price density (SPD) when there exists a continuum of states of nature, summarizes all the relevant information about preferences for asset pricing purposes. In this respect, the SPD generalizes the notion of Arrow-Debreu asset prices—which give for each state of nature the price of a security paying one dollar in that specific state and nothing in any other state.

There exists a close relation between option prices and SPDs. From (8), the price a call option, $\pi_t = \pi_t(S_T, K)$, with terminal payoff $\max[0, S_T - K]$ is given as

$$\pi_t(S_T, K) = B(t, T) \int_0^\infty \max[0, S_T - K] f^*_t(S_T) \, dS_T,$$

where $f^*_t(\cdot)$ denotes the relevant multi-period SPD. Note that, for sake of notational simplicity, its dependence on the horizon $T$ is not made explicitly. Conversely, following Banz and Miller (1978) and Breeden and Litzenberger (1978), one can recover the SPD from the observation of a cross-section of option prices at different strike prices. It is actually sufficient to differentiate twice the above formulas with respect to the strike price $K$ to get:

$$f^*_t(S_T) = \frac{1}{B(t, T)} \left[ \frac{\partial^2 \pi_t(S_T, K)}{\partial K^2} \right]_{|K=S_T}.$$

Recall that under the assumptions of the BS model, stock returns are lognormally distributed under the risk-neutral probability measure. Jackwerth and Rubinstein (1996), however, provide nonparametric evidence showing that the shape of the SPD implied by option prices is not consistent with lognormality. In particular, they find that the SPD is left-skewed and leptokurtic since the stock market crash of 1987.

The failures of the benchmark BS model to describe the structure of observed option prices can equivalently be characterized by so-called implicit or implied volatilities. Given an observed option price, $\pi^{obs}_t$, a unique BS implied volatility, $\sigma^*_t$, can be found such that

$$\pi^{obs}_t = BS[S_t, K, \sigma^*_t].$$

If observed market option prices were indeed given by the constant-volatility BS model, then the implied volatilities thus extracted from a set of options written on the same asset
and differing only by their strike prices would all be identical in value. In reality, the implied volatilities, when plotted against a measure of option moneyness, result in curves thus indicating violations of the BS assumptions. These empirical biases of the BS model have been dubbed the smile effect with reference to symmetric U-shaped implied volatility curves, although distorted smiles in the form of smirks or frowns are frequently observed. In Figure 1, we graph several volatility curves for options on the S&P 500 index on selected dates to show the forms most frequently observed.

In Figure 1, we graph several volatility curves for options on the S&P 500 index on selected dates to show the forms most frequently observed.

\[
\sigma^*_t(x_t) = \sigma^*_t(-x_t),
\]

for all \(x_t\), with the minimum occurring at the money \((x_t = 0)\).

Renault and Touzi (1996) and Renault (1997) show that if option prices are given by

\[
\pi_t = \tilde{E}_t \left[ BS[S_t, K, \sigma^2_{t,T}] \right],
\]

where the expectation is applied over some heterogeneity distribution of the volatility parameter, \(\sigma^2_{t,T}\), then the resulting volatility smile is necessarily symmetric. In that case, any pair of strike prices chosen so that their geometric mean is the current forward price yield the same option price. The Hull and White (1987) stochastic volatility model is a
well-known option pricing formula in the form of expectations over BS prices as in Equation (11). By extension, asymmetric features of the volatility smile are often interpreted as revealing asymmetry of the risk-neutral distribution of the underlying asset log-return. The mixture model of Section 3 is particularly well suited to accommodate such asymmetries.

For sake of notational simplicity, assume that interest-rate risk can be neglected, so that \( B^N(t, T) = B(t, T) \). Such an assumption is generally considered as reasonable for sufficiently short maturity options on equity. When applied to the pricing of a European call option with payoff \( h_T = h(S_T) = \max[0, S_T - K] \), the mixture model gives:

\[
\pi_t(h_T) = E_t \left\{ E_t [B(t, T) \max[0, S_T^N Q(t, T) - K] \mid U_{1}^{T}] \right\},
\]

(12)

where \( U_{1}^{T} \) means the set of all relevant state variable values \( U_{t+h}, h = 1, \ldots, T - t \), over the lifetime of the option. Note however that to derive (11) from (6), two additional technical assumptions are implicitly maintained. First, the pairs \( \log(m_{t+h}), \log(S_{t+h}/S_{t+h-1}) \), \( h = 1, \ldots, T - t \), are assumed to be serially independent given the path \( U_{t}^{T} \) of state variables. Second, they do not Granger-cause the state variables. In other words, latent state variables which define the dynamic mixture components are assumed to be exogenous and to summarize all the relevant dynamics. These assumptions are conformable to the tradition of both stochastic volatility models and Markov-switching regime models. Then, conditional on \( U_{t}^{T} \), we are in a BS risk-neutral world. Therefore, we deduce the following Generalized Black-Scholes (GBS) option pricing formula:

\[
\pi_t(h_T) = E_t \left\{ BS(S_t Q(t, T), K, \sigma^2(U_{t}^{T})) \right\}
\]

(13)

where, inside the standard BS formula, the current value of the stock price \( S_t \) has been replaced by \( S_t Q(t, T) \) and the constant volatility parameter, \( \sigma^2 \), has been replaced by the stochastic volatility term \( \sigma^2(U_{t}^{T}) = Var_t[\log(S_T/S_t) \mid U_{t}^{T}] \). The expectations operator in (13) is with respect to the joint probability distribution of \( Q(t, T) \) and \( \sigma^2(U_{t}^{T}) \) conditional on the time-\( t \) information set, \( J_t \).

Pricing formulas similar to that in (13) were derived by Romano and Touzi (1997) in the context of risk-neutral continuous-time models of stochastic volatility with leverage, as in Heston (1993), and by Fouque, Papanicolaou and Sircar (2000), where the random variable \( Q(t, T) \) plays a similar role. However, the GBS formula in (13) is even more convenient for empirical pricing since the expectations operator is considered with respect to the objective probability measure instead of the aforementioned formulas involving an equivalent martingale or risk-neutral probability measure. Indeed, Willard (1997) notes in
the context of the Heston (1993) model, that option pricing formulas viewed as expectations of BS prices (and associated greeks derived as expectations of BS greeks\(^5\)) are particularly well suited for Monte Carlo simulation of option prices, since they become a conditional Monte Carlo where only the state variable process needs to be simulated.

The GBS formula is particularly useful to interpret the observed asymmetric shapes of the volatility smile. Since the stock market crash of 1987, observed volatility smiles are often skewed and even reversed in the form of smirks and frowns. In the context of GBS pricing, these asymmetries are explained by the fact that \( Q(t, T) \) is genuinely random, even though it is equal to one in expectation. Therefore, the asymmetries are due to a mixture effect either in the distribution of the returns \( E_t[S_T/S_t \mid U^T_t] \neq E_t[S_T/S_t] \) or in its covariation with the SDF \( \text{Cov}_t[\log(m_{t,T}), \log(S_T/S_t \mid U^T_t)] \), which creates various kinds of leverage effects. We discuss further these leverage effects in Section 6 below in the setting of an intertemporal equilibrium model with state variables.

One way to understand the strength of these leverage or instantaneous causality effects is to recall that the BS formula is convex with respect to the current price of the underlying asset. Jensen’s inequality and the fact that \( E_t[Q(t, T)] = 1 \) implies that the GBS price should be greater than the price in (11) corresponding to symmetric volatility smiles, unless the correlation between \( Q(t, T) \) and \( \sigma^2(U^T_t) \) would reverse the Jensen effect. Therefore, GBS pricing should be equivalent to replacing the current asset price by a greater (deterministic) value. Manaster and Rendleman (1982), Longstaff (1995), and Garcia, Luger, and Renault (2002) provide some evidence of such implied index values often greater than the current spot value. Renault (1997) shows through simulations in a Hull and White (1987) model that even very small differences between the implied stock price and its value used to compute implied volatilities will produce severe asymmetries corresponding to widely observed smirks and frowns.

5. Nonparametric Assessments of Revealed Preferences

The SPD contains valuable information about the preferences of the representative investor since it corresponds to the Arrow-Debreu prices in the continuum-of-states case. In partic-

\(^5\)The greeks are a set of factor sensitivities used extensively by practitioners to quantify the exposures of portfolios that contain options. Each measures how the portfolio’s market value should respond to a change in one of the following variables: (i) an underlying stock, (ii) an implied volatility, (iii) an applicable interest rate, or (iv) time. See Hull (2000, chap. 13).
ular, the ratio of the SPD to the conditional objective probability density is proportional to the SDF as explained in Section 2:

\[
f^*_t(S_T)/f_t(S_T) = m_{t,T}[B(t,T)]^{-1}.
\]

As in the previous section, the notation neglects the fact that the multi-period (forward) SPD may depend on the horizon \( T \). By focusing on the SDF provided by a time separable utility function (see Section 6 below for an extension) and assuming that \( S_T \)—the value of the index upon maturity of the option—approximates aggregate wealth and is proportional to aggregate consumption, we know that \( m_{t,T} \) should be proportional to \( u'(S_T) \). Therefore, taking log-derivatives on the above relations between SPDs and SDF, we deduce that a measure of absolute risk aversion is given by

\[
\rho_t(S_T) = S_T \left( \frac{f'_t(S_T)}{f_t(S_T)} - \frac{f''_t(S_T)}{f'_t(S_T)} \right),
\]

where \( f'_t(S_T) \) and \( f''_t(S_T) \) denote the derivatives of the conditional objective probability density and the SPD, respectively. Aït-Sahalia and Lo (2000), Jackwerth (2000), and Rosenberg and Engle (2002) follow this approach to extract preferences based on nonparametric techniques.

The strategy proposed by Aït-Sahalia and Lo (2000) begins by estimating a nonparametric regression of BS implied volatilities on a set of explanatory variables. In particular, Aït-Sahalia and Lo use a nonparametric estimator of the expectation of volatility given the information available on the underlying stock price \( S_t \) (or the futures price \( F_{t,\tau_i} = S_t e^{(r_t - \delta_t)\tau_i} \), where \( r \) and \( \delta \) denote the interest rate and the dividend rate, respectively), the strike price \( K_i \), and the time to maturity \( \tau_i \) associated with \( n \) traded options:

\[
\hat{\sigma}(F_{t,\tau_i}, K_i, \tau_i) = \frac{\sum_{i=1}^{n} k_F \left( \frac{F_{t,\tau_i} - F_{t,\tau_i}}{h_F} \right) k_K \left( \frac{K-K_i}{h_K} \right) k_{\tau} \left( \frac{\tau-\tau_i}{h_{\tau}} \right) \sigma_i^*}{\sum_{i=1}^{n} k_F \left( \frac{F_{t,\tau_i} - F_{t,\tau_i}}{h_F} \right) k_K \left( \frac{K-K_i}{h_K} \right) k_{\tau} \left( \frac{\tau-\tau_i}{h_{\tau}} \right)},
\]

where the multivariate kernel is formed as a product of three univariate kernels \( k_F, k_K, \) and \( k_{\tau} \) each with their own bandwidth value with respect to the three variables of interest, and where \( \sigma_i^* \) is the BS volatility implied by the observed price of option \( i \). Once a smoothed implied volatility function is obtained, a call pricing function can then be estimated as

\[
\hat{\pi}(S_t, K, \tau, r_{t,\tau}, \delta_{t,\tau}) = \pi_{BS}(F_{t,\tau}, K, \tau, r_{t,\tau}, \hat{\sigma}(F_{t,\tau}, K, \tau)).
\]
From this function, one can also obtain estimators for the option’s delta and the SPD by taking the appropriate partial derivatives:

\[ \hat{\Delta}_t = \frac{\partial \hat{\pi}(S_t, K, \tau, r_t, \delta_t)}{\partial S_t}, \quad (17) \]

\[ \hat{f}^*_t(S_T) = e^{r_{t+\tau}} \left[ \frac{\partial^2 \hat{\pi}(S_t, K, \tau, r_t, \delta_t)}{\partial K^2} \right] _{K = S_T}. \quad (18) \]

In order to estimate the objective probability density \( f \), Aït-Sahalia and Lo (2000) and Jackwerth (2000) use standard kernel density estimation techniques applied to the historical time series of realized returns. When applied to S&P 500 index European option price data, both Aït-Sahalia and Lo (2000) and Jackwerth (2000) obtain risk aversion functions that are not nearly as well behaved as those suggested by economic theory. Indeed, standard assumptions imply positive and monotonically downward sloping risk aversion functions, while those reported by Aït-Sahalia and Lo (2000) and Jackwerth (2000) eventually increase with wealth. A number of possible reasons, including purely statistical ones like bandwidth choice, could explain these findings.

It is important, however, to note the unconditional nature of this nonparametric methodology. The methodology assumes that the SPD is a fixed function of the vector of explanatory variables over the sample period. The estimates so obtained are best interpreted as a measure of the average SPD over the sample. These estimated SPDs are thus limited in their ability to price and hedge assets in real-time trading, since by their very nature they cannot accommodate a state-dependent SPD.

The unconditional approach described above should be contrasted with the empirical pricing kernel estimation method proposed by Rosenberg and Engle (2002), where the SDF is projected onto the payoffs of a derivative security, \( g_i(r_{t+1}) \), that depends on the return, \( r_{t+1} \), to the underlying asset. This method is based on an estimated asset return probability density, \( \hat{f}_i(r_{t+1}) \), for which Rosenberg and Engle use an asymmetric GARCH specification with an empirical innovation density. The projected pricing kernel, \( m^*(r_{t+1}; \hat{\theta}_t) \), is defined through the solution of the following optimization problem:

\[ \min_{\theta_t} \sum_{i=1}^{L} \left[ p_{i,t} - \hat{p}_{i,t}(\theta_t) \right]^2, \]

where \( L \) represents the number of asset prices, and

\[ \hat{p}_{i,t}(\theta_t) = \int_{-\infty}^{\infty} m^*(r_{t+1}; \theta_t) g_i(r_{t+1}) \hat{f}_i(r_{t+1}) \, dr_{t+1}. \]
Rosenberg and Engle consider two parametric specifications of the pricing kernel. The first is a power function of the underlying asset’s return of the form
\[ m^*(r_{t+1}; \theta_t) = \theta_{0,t} (r_{t+1})^{-\theta_{1,t}}, \]
while the second—a generalized Chebyshev polynomial—allows more flexibility in the shape of the pricing kernel. When applied to S&P 500 index returns and option prices from 1991 to 1995, they find that the polynomial specification fits option prices better, but that the power specification is superior in terms of hedging.

The analysis in Rosenberg and Engle (2002) also shows the empirical risk aversion parameter varying over time, with counter-cyclical movements. To give a more structural interpretation to their empirical risk aversion parameter, suppose that the conditional distribution of \( (\log m^*_{t+1}, \log r_{t+1}) \) given some relevant state variable, \( U_{t+1} \), is bivariate normal:
\[
N \left[ \begin{pmatrix} \mu_m(U_{t+1}^t) \\ \mu_r(U_{t+1}^t) \end{pmatrix}, \begin{pmatrix} \sigma^2_m(U_{t+1}^t) & \sigma_{mr}(U_{t+1}^t) \\ \sigma_{mr}(U_{t+1}^t) & \sigma^2_r(U_{t+1}^t) \end{pmatrix} \right].
\]

The conditional expectation of \( \log m_{t+1} \) given \( \log r_{t+1} \) and \( U_{t+1}^t \), is given by the linear form:
\[
E[\log m_{t+1} | \log r_{t+1}, U_{t+1}^t] = \mu_m(U_{t+1}^t) + \frac{\sigma_{mr}(U_{t+1}^t)}{\sigma^2_m(U_{t+1}^t)} (\log r_{t+1} - \mu_r(U_{t+1}^t)).
\]
An empirical pricing kernel specification based on \( E_t[\log m_{t+1} | \log r_{t+1}] = E_t\{E_t[\log m_{t+1} | \log r_{t+1}, U_{t+1}]\} \), where the effects of the state variable are properly filtered out, would in this case yield time-varying coefficients provided of course that the state variable \( U_{t+1} \) features some predictability. In other words, the coefficients in a reduced form, \( \log m^*_{t+1} = \log \theta_{0,t} - \theta_{1,t} \log r_{t+1} \), would display time variation at business-cycle frequencies insofar as \( \mu_m(U_{t+1}^t) \), \( \sigma_{mr}(U_{t+1}^t) \), and \( \sigma^2_m(U_{t+1}^t) \) are predictable from the current state of the economy.

The general conclusion is that empirical pricing kernels computed without a proper account of all the relevant state variables that enter the SPD are likely to mislead interpretations about the intertemporal equilibrium structure of preferences. This point has also been made by Chabi-Yo, Garcia, and Renault (2004), where an economy characterized by regime changes either in fundamentals or preferences—the two possible sources of state dependence in the SPD—is considered. They show that an application of Jackwerth’s non-parametric methodology leads to similar negative estimates of the risk aversion function in some states of wealth even though the risk aversion functions are consistent with economic theory within each regime. This is illustrated in the next section in the context of an intertemporal equilibrium asset pricing model with state variables.
6. An Intertemporal Equilibrium Model with State Variables

The previous section emphasized the potential importance of investors’ preferences for option prices but the question of knowing if option prices are compatible with reasonable preferences remains largely unanswered. One possible explanation for the divergence between the objective and the risk-neutral distributions is the existence of time-varying risk premiums. Pan (2002) estimates a jump-diffusion model proposed by Bates (2000) and investigates how volatility and jump risks are priced in S&P 500 index options. Based on a joint time series of the spot asset price and of one at-the-money option, Pan shows that the addition of both volatility and jump risk premiums allows to fit well the joint time series of spot asset and option price data. The model can explain well the changing shapes of the implied volatility curves over time and the skewed patterns are largely attributable to investors’ aversion to jump risks. However, it is not clear how this non-arbitrage continuous-time model relates to the preferences of a representative agent since in this approach investors may have different risk attitudes towards the diffusive return shocks, volatility shocks, and jump risks.

In this section we revisit the role of preferences and state variables in option pricing from an equilibrium point of view. We consider a structural utility-based option pricing model as in Garcia, Luger, and Renault (2003) (see also Garcia and Renault 1998b), where stochastic volatility and jump features are governed by a latent state variable process. The advantage of the equilibrium framework is that here stock returns are determined in equilibrium rather than being modeled directly as a stochastic process. Therefore, the assumptions about the joint dynamics of asset returns, SDF, and state variables that have been maintained so far are now justified by assumptions about the fundamentals of the economy. Moreover, our equilibrium model is based on a more general utility function that retains the attractive features of power utility but disentangles the respective roles of discounting, risk aversion, and intertemporal substitution.

We adopt the recursive utility framework proposed by Epstein and Zin (1989). Many identical infinitely lived investors maximize their lifetime utility and receive each period an endowment of a single nonstorable good. Their recursive utility function is of the form:

\[ V_t = W(C_t, \mu_t), \]

where \( W \) is an aggregator function that combines current consumption, \( C_t \), with \( \mu_t = \)
\( \mu(\tilde{V}_{t+1} \mid J_t) \), a certainty equivalent of random future utility \( \tilde{V}_{t+1} \), given \( J_t \) the information available to the investors at time \( t \), to obtain the current period lifetime utility \( V_t \). Following Kreps and Porteus (1978), Epstein and Zin (1989) propose the following CES function as aggregator function:

\[
V_t = [C_t^\rho + \beta \mu_t^\rho]^\frac{1}{\rho}.
\]

The way investors form the certainty equivalent of random future utility is based on their risk preferences, which are assumed to be isoelastic; i.e., \( \mu_t^\alpha = E[\tilde{V}_{t+1}^\alpha \mid J_t] \), where \( \alpha \leq 1 \) is the risk aversion parameter (1 – \( \alpha \) is the Arrow-Pratt measure of relative risk aversion).

Given these preferences, the following Euler condition must be valid for any asset \( j \) if an investor maximizes his lifetime utility (see Epstein and Zin 1989):

\[
E[\beta^\gamma (\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} R_{j,t+1} \mid J_t] = 1, \tag{19}
\]

where \( M_{t+1} \) represents the return on the market portfolio, \( R_{j,t+1} \) the return on any asset \( j \), and \( \gamma = \alpha/\rho \). The parameter \( \rho \) is associated with intertemporal substitution, since the elasticity of intertemporal substitution is \( 1/(1-\rho) \). The position of \( \alpha \) with respect to \( \rho \) determines whether the investor has a preference towards early resolution of uncertainty (\( \alpha < \rho \)) or late resolution of uncertainty (\( \alpha > \rho \)).

Since the market portfolio price, say \( P_t^M \) at time \( t \), is determined in equilibrium, it should also verify the first-order condition in (19); i.e.,

\[
E[\beta^\gamma (\frac{C_{t+1}}{C_t})^{\gamma(\rho-1)} M_{t+1}^{\gamma-1} \mid J_t] = 1.
\]

In this model, the payoff of the market portfolio at time \( t \) is the total endowment of the economy, \( C_t \). Therefore the return on the market portfolio, \( M_{t+1} \), can be written as

\[
M_{t+1} = \frac{P_{t+1}^M + C_{t+1}}{P_t^M}.
\]

Under some regularity assumptions, the rates of return on the market portfolio and the stock can be described by

\[
\log M_{t+1} = \log \frac{\lambda_{t+1} + 1}{\lambda_t} + \log \frac{C_{t+1}}{C_t} \tag{20}
\]

and

\[
\log R_{t+1} = \log \frac{S_{t+1} + D_{t+1}}{S_t} = \log \frac{\varphi_{t+1} + 1}{\varphi_t} + \log \frac{D_{t+1}}{D_t}. \tag{21}
\]
where $\lambda_t = P_t^M/C_t$ and $\phi_t = S_t/D_t$ are the price-dividend ratios on the market portfolio and the stock, respectively.

The dynamics in (20) and (21) are determined by the joint probability distribution of the stochastic process $(X_t, Y_t, J_t)$ with $X_t = \log \frac{C_t}{C_{t-1}}$ and $Y_t = \log \frac{D_t}{D_{t-1}}$. We shall define these dynamics through a stationary vector process of state variables $U_t$ such that

$$J_t = \vee_{\tau \leq t} [X_\tau, Y_\tau, U_\tau].$$

Although the structure of the economy, and in particular the probability distribution of the state variable are common knowledge among the investors, the realizations of the state variables are assumed as in Section 3 not to be instantaneously observable by the investors at the beginning of period $t$. It is only at the end of the period that investors learn the realized value.

We want the state variable to be exogenous and stationary, and to subsume all temporal links between the variables of interest, $(X_t, Y_t)$. This framework has already been used in asset pricing models; see Cecchetti, Lam, and Mark (1990, 1993), Bonomo and Garcia (1994a, b, 1996) and Amin and Ng (1993). We achieve this through the following assumptions.

**Assumption 6.1.** The pairs $(X_t, Y_t)_{1 \leq t \leq T}, t = 1, \ldots, T$, are mutually independent given $U_{T}^T = (U_t)_{1 \leq t \leq T}$.

**Assumption 6.2.** The fundamentals, $(X, Y)$, do not cause the state variables, $U$, in the Granger sense or equivalently, given Assumption 6.1, the conditional probability distribution of $(X_t, Y_t)$ given $U_{T}^T = (U_t)_{1 \leq t \leq T}$ coincides, for $t = 1, \ldots, T$, with the conditional probability distribution given $U_{t}^T = (U_\tau)_{1 \leq \tau \leq t}$.

**Assumption 6.3.** The conditional probability distribution of $(X_{t+1}, Y_{t+1}, U_{t+1})$ given $U_t^T$ only depends upon $U_t$.

The fundamentals are assumed to be normally distributed conditional on the state variable.

**Assumption 6.4.**

$$\left( \begin{array}{c} X_{t+1} \\ Y_{t+1} \end{array} \right) \mid U_{t}^{t+1} \sim N \left( \begin{array}{c} m_{X_{t+1}} \\ m_{Y_{t+1}} \end{array} \right), \left[ \begin{array}{cc} \sigma^2_{X_{t+1}} & \sigma_{XY_{t+1}} \\ \sigma_{XY_{t+1}} & \sigma^2_{Y_{t+1}} \end{array} \right].$$
where \( m_{X_{t+1}} = m_X(U_{t+1}) \), \( m_{Y_{t+1}} = m_Y(U_{t+1}) \), \( \sigma_{X_{t+1}}^2 = \sigma_X^2(U_{t+1}) \), \( \sigma_{XY_{t+1}} = \sigma_{XY}(U_{t+1}) \), \( \sigma_{Y_{t+1}}^2 = \sigma_Y^2(U_{t+1}) \).

Given this intertemporal framework with latent state variables, Garcia, Luger, and Renault (2003) show that the price of a European call option is given by the following GBS formula:

\[
\pi_t = E_t \left\{ S_t Q_{XY}(t, T) \Phi(d_1) - K \tilde{B}(t, T) \Phi(d_2) \right\},
\]

where

\[
d_1 = \log \left( \frac{S_t Q_{XY}(t, T)}{K \tilde{B}(t, T)} \right) + \frac{1}{2} \left( \sum_{\tau=t+1}^{T} \sigma_{Y_{\tau}}^2 \right)^{1/2} \text{ and } d_2 = d_1 - \left( \sum_{\tau=t+1}^{T} \sigma_{Y_{\tau}}^2 \right)^{1/2}.
\]

The terms \( Q_{XY}(t, T) \) and \( \tilde{B}(t, T) \) in (22) are stochastic factors that determine the equilibrium prices of the stock and the bond. The time-\( t \) price of a bond which delivers one unit of the good at time \( T \) is given by \( B(t, T) = E_t[\tilde{B}(t, T)] \) and stocks are compensated in equilibrium so that \( E_t[Q_{XY}(t, T)] = 1 \). These prices depend on the means and variance-covariance functions, which in turn depend on the latent state variable and the preference parameters. It should be noticed that if \( Q_{XY}(t, T) = 1 \) and \( \tilde{B}(t, T) = \prod_{\tau=t}^{T-1} B(\tau, \tau + 1) \), the option price in (22) is nothing but the conditional expectation of the BS price, where the expectation is computed with respect to the joint probability distribution of the rolling-over interest rate \( r_{t,T} = -\sum_{\tau=t}^{T-1} \log B(\tau, \tau + 1) \) and the cumulated volatility \( \sigma_{t,T} = \sqrt{\sum_{\tau=t+1}^{T} \sigma_{Y_{\tau}}^2} \).

This framework nests three well-known models. First, the most basic ones, the Black and Scholes (1973) and Merton (1973) formulas, when interest rates and volatility are deterministic. Second, the Hull and White (1987) stochastic volatility extension, since \( \sigma_{t,T}^2 = Var \left[ \log \frac{S_T}{S_t} \mid U_t^T \right] \) corresponds to the cumulated volatility \( \int_t^T \sigma_u^2 du \) in the Hull-White continuous-time setting. Third, the formula allows for stochastic interest rates as in Turnbull and Milne (1991) and Amin and Jarrow (1992). However, the usefulness of the general formula in (22) comes above all from the fact that it offers an explicit characterization of instances where the preference-free paradigm cannot be maintained.

Usually, preference-free option pricing is underpinned by the absence of arbitrage in a complete market setting. However, our equilibrium-based option pricing formula does not preclude incompleteness and points out in which cases this incompleteness will

\[\text{We refer here to a BS option pricing formula where dividend flows arrive during the lifetime of the option and are accounted for in the definition of the risk-neutral probability, while the option payoff does not include dividends.}\]
invalidate the preference-free paradigm; i.e., when the conditions $Q_{XY}(t, T) = 1$ and $\tilde{B}(t, T) = \prod_{\tau = t}^{T-1} B(\tau, \tau + 1)$ are not fulfilled. In that case, preference parameters appear explicitly in the option pricing formula through $\tilde{B}(t, T)$ and $Q_{XY}(t, T)$.

It is worth noting that our results of equivalence between preference-free option pricing and no instantaneous causality between state variables and asset returns are consistent with another strand of the option pricing literature, namely GARCH option pricing. Duan (1995) derives it in an equilibrium framework, but Kallsen and Taqqu (1998) show that it could be obtained with an arbitrage argument. Their idea is to complete the markets by inserting the discrete-time model into a continuous time one, where conditional variance is constant between two integer dates. They show that such a continuous-time embedding makes possible arbitrage pricing which is per se preference free. It is then clear that preference-free option pricing is incompatible with the presence of an instantaneous causality effect, since it is such an effect that prevents the embedding used by Kallsen and Taqqu (1998).

Garcia, Luger, and Renault (2001) show that this intertemporal option pricing model explains many of the observed asymmetries of implied volatility smiles. They illustrate the types of asymmetries that can be caused by leverage effects through both the consumption and the dividend processes. They also discuss how the smile varies depending on the values of the preference parameters.

In Garcia, Luger, and Renault (2003), the same model is estimated with S&P 500 index option prices from 1991 to 1995. Table 1 reports the average values of the preference parameters that we obtained in each of the five years and over the five-year period. Looking first at the GBS model, we can say that the estimates of the coefficient of relative risk aversion (CRRA) and elasticity of intertemporal substitution (EIS) appear reasonable. Over the five-year period, the CRRA equals 0.6838 on average and the EIS averages 0.8532. These results are quite intuitive since one generally expects that the inverse of EIS should be greater than CRRA, as Weil (1989) emphasizes. As the yearly means and standard errors indicate, the values obtained are remarkably stable over time—a reassuring fact for a structural model with representative investor. It is interesting to compare these estimates with the values obtained when the parameter $\gamma$ is constrained to equal 1 (the expected utility case). Similarly to what was obtained with stock returns series in various studies aimed at solving the equity premium puzzle, we obtain a high average value of 7.159 for the CRRA with a standard deviation of 4.826. Over the same time period, Rosenberg

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7 Heston and Nandi (2000) point out that the GARCH option pricing model of Duan (1995) is valid if and only if BS is valid over one period.
and Engle (2002) also find an empirical risk aversion of 7.36 with a power utility function defined over wealth (measured by the S&P 500 index), based on S&P 500 option price data. Therefore, relaxing the constraint $\gamma = 1$ allows for a more reasonable value for the EIS. The value found for $\beta$ in the expected utility case is somewhat low (0.88 on average), while it appears more reasonable (0.94 on average) when $\gamma$ is unconstrained.

These results, in conjunction with further evidence on the out-of-sample pricing performance of the generalized non-expected utility option pricing formula found in Garcia, Luger, and Renault (2003), clearly show that preferences matter for option pricing and that option prices help distinguish between expected and non-expected utility models. We also confirm the informativeness of option prices about preference parameters in a simulation experiment. In contrast to Jackwerth (2000) who infers theoretically implausible risk-aversion functions, the estimates we obtain for the preference parameters are economically plausible and meaningful. It should be emphasized that preference parameters enter consistently in the equilibrium pricing of all assets in our model. We illustrate the risk aversion puzzle in the context of this equilibrium model in Figure 2. The graph on the left hand side reveals that the unconditional pricing kernel increases in the center wealth states (over the range 0.9-1.1), while in the right-hand graph, also around the center wealth states, the unconditional absolute risk aversion function is negative as in Jackwerth (2000). Within each regime, however, the conditional pricing kernel and absolute risk aversion function across wealth states are perfectly decreasing functions of aggregate wealth: the puzzles disappear.

7. Conclusion

In this viewpoint we have emphasized the interplay between preferences and option pricing. In particular, we have specified the statistical assumptions that are needed to obtain preference-free option pricing formulas. By the same token we have characterized the covariance or leverage effects which reintroduce preferences in option prices. In an equilibrium
framework, we have clearly identified the role of preferences in an option pricing formula which is a natural generalization of the BS formula. In approaches based on the absence of arbitrage, these preferences are hidden in risk premiums and it is harder to account for the links they impose between the premiums associated with the numerous sources of risk. Researchers often treat these risk premiums as free parameters and manage to capture some empirical facts, but a deeper economic explanation is lacking. The extraction of preferences from option prices using nonparametric methods is even more problematic. The puzzles associated with this literature often come from the fact that state variables have been omitted in the analysis. Contrary to what was found in the empirical literature based on stock returns, the preference parameters estimated from option prices appear reasonable and consistent with economic theory.

References


Table 1: Yearly Means and Standard Errors of Daily Estimated Preference Parameters from S&P 500 Option and Stock Price Data.

<table>
<thead>
<tr>
<th></th>
<th>GBS Model</th>
<th>Expected Utility Model</th>
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<tbody>
<tr>
<td></td>
<td>$\rho$</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>1991</td>
<td>-0.2048 (0.0904)</td>
<td>-1.6637 (0.9144)</td>
</tr>
<tr>
<td>1992</td>
<td>-0.0936 (0.0400)</td>
<td>-1.9975 (0.4171)</td>
</tr>
<tr>
<td>1993</td>
<td>-0.2007 (0.0737)</td>
<td>-2.4294 (1.1218)</td>
</tr>
<tr>
<td>1994</td>
<td>-0.2110 (0.1211)</td>
<td>-1.7369 (0.6011)</td>
</tr>
<tr>
<td>1995</td>
<td>-0.1963 (0.1504)</td>
<td>-1.8744 (0.7700)</td>
</tr>
<tr>
<td>1991-1995</td>
<td>-0.1812 (0.1114)</td>
<td>-1.9406 (0.8458)</td>
</tr>
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Note: The estimation is based on an exact method of moments described in Garcia, Luger, and Renault (2003). CRRA denotes the coefficient of relative risk aversion, EIS the elasticity of intertemporal substitution.
Examples of implied volatility curves inferred from S&P 500 call option Prices

15 day S&P 500 call option: 89-01-05

18 day S&P 500 call option: 94-01-03

49 day S&P 500 call option: 92-05-01

79 day S&P 500 call option: 92-07-01

11 day S&P 500 call option: 89-01-09

30 day S&P 500 call option: 90-05-16

Figure 1
Figure 2: Absolute Risk Aversion (ARA) and Pricing Kernel functions with state dependence in fundamentals. The preference parameters are: $\beta = 0.96$, $\alpha = -1.53$, $\rho = -0.8$. The regime probabilities are: $p_{11} = 0.45$, $p_{00} = 0.4$. For the economic fundamentals, the means of the consumption growth rate are $\mu_{X_{t+1}} = (0.001, 0.005)$, and the corresponding standard deviations $\sigma_{X_{t+1}} = (0.003, 0.006)$. For the dividend rate, the parameters are $\mu_{Y_{t+1}} = (0, 0.003)$, $\sigma_{Y_{t+1}} = (0.065, 0.26)$. The correlation coefficient between consumption and dividends is $-0.345$. The number of options used is 50. The number of wealth states is $n = 170$. The left-hand panel contains the conditional and unconditional pricing kernel functions across wealth states. The right-hand panel contains the conditional and unconditional ARA functions across wealth states. The conditional ARA (pricing kernel) function is the ARA (pricing kernel) function computed within each regime. The unconditional ARA (pricing kernel) function is the ARA (pricing kernel) function computed when regimes are not observed.